Two Network Models with Very Heavy Tails

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1. Introduction: Very Heavy Tails.

Attempts to explain network self-similarity focus on heavy tailed transmission times of sources sending data to one or more servers:

$$P[\text{On-period duration } > x] \approx x^{-\beta}.$$

Reasons for assuming $1 < \beta < 2$:

- (Applied) Willinger et al (Bellcore) analyzed 700+ source destination pairs, and estimated the tail parameter of *on-periods*. Value usually in the range $(1, 2)$.
- (Theoretical) Mathematical analysis has been based on renewal theory. Without a finite mean, stationary versions of renewal processes do not exist and (uncontrolled) buffer content stochastic processes would not be stable.

BUT

Need for the case $0 < \beta < 1$:

- BU study (Crovella, Bestavros, ...) of file sizes downloaded in www session over 4 months in 2 labs. In November, 1994 in room 272: $\beta \approx .66$.
- Calgary study of file lengths downloaded from various servers found β's in the range of 0.4 to 0.6.

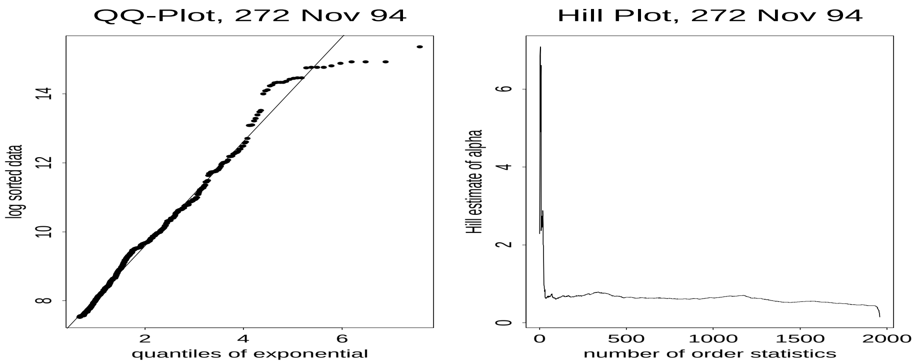

Figure 1: BU File Lengths, Nov 1994, Room 272

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1.1. Summary of the difficulties with the case $0 < \beta < 1$.

- Models will not be stable in the conventional senses; normalization necessary to keep control.
- Models will not have stationary versions.
- Some common performance measures which are expressed in terms of moments, may not be applicable.
- Nervousness about models where moments do not exist.
- Confusion between concepts of unbounded support and infinite moments. (Normal, exponential, gamma, weibull, ... have unbounded support.)

2. Infinite source Poisson model=M/G/∞ input model

Notation and concepts:

$$\{\Gamma_n, n \geq 1\} = \text{times of session initiations; homogeneous Poisson points on } [0,\infty) \text{ with rate } \lambda.$$

$$\{T_n\} = \text{session durations; iid non-negative rv's with common distribution } G(x) \text{ satisfying}$$

$$\bar{G}(x) \sim x^{-\beta} L(x), \quad 0 < \beta < 1;$$

$$m(t) = \int_0^t \bar{G}(s)ds \sim ct^{1-\beta} L(t) \uparrow \infty; \text{ truncated mean function,}$$

$$M(t) = \text{number of active sessions at } t,$$

$$= \sum_{n=1}^{\infty} 1_{[\Gamma_n \leq t < \Gamma_n + T_n]}; \quad E(M(t)) = m(t);$$

$$= \# \text{ of servers in M/G/}\infty \text{ telephone model.}$$

$$A(t) = \int_0^t M(s)ds,$$

$$= \text{cumulative work inputted in } [0,t] \text{ assuming unit rate input.}$$

$$r = \text{server work rate},$$

$$X(t) = \text{content process},$$

$$dX(t) = dA(t) - r 1_{[X(t)>0]} dt$$

$$\tau(\gamma) = \inf\{t > 0 : X(t) \geq \gamma\}$$

3. Very Heavy Tails and Gaussian Approximations (Rootzen)

3.1. First order behavior.

As $T \to \infty$, in probability,

$$\begin{aligned} M(T) &\sim m(T) = cT^{1-\beta}L(T), \\ A(T) &\sim X(T) \sim cTm(T) = cT^{2-\beta}L(T), \end{aligned}$$

and as $\gamma \to \infty$

$$\tau(\gamma) \sim (2-\beta)^{1/(2-\beta)}V^{\leftarrow}(\gamma)$$

where

$$V(T) = Tm(T) = cT^{2-\beta}L(T)$$

so

$$V^{\leftarrow}(\gamma) = c\gamma^{1/(2-\beta)}L'(\gamma).$$

3.2. Second order behavior; Gaussian approximation.

We have M, A, X asymptotically Gaussian.

3.2.1. Approximations for $M(\cdot)$.

As $T \to \infty$, in $D[0, \infty)$,

$$G_T(\cdot) = \frac{M(T\cdot) - m(T\cdot)}{\sqrt{m(T)}} \Rightarrow G(\cdot).$$

Properties of the limit $G(\cdot)$:

- $G(\cdot)$ is a zero mean, continuous path, self-similar Gaussian process.
- The covariance function $C(s, t)$ of G is

$$C(s, t) = (s \vee t)^{1-\beta} - |t - s|^{1-\beta}, \quad 0 \le s \le t.$$

BUT

Note

- $G(\cdot)$ is not stationary,
- $G(\cdot)$ does not have stationary increments
- $G(\cdot)$ is not a fractional Brownian motion.

3.2.2. Approximations for $A(\cdot)$.

Integrated version of results for M: As $T \to \infty$

$$\hat{G}_T(\cdot) := \frac{A(T\cdot) - \int_0^{T(\cdot)} m(s)ds}{T\sqrt{m(T)}} \Rightarrow \hat{G}(\cdot) = \int_0^{(\cdot)} G(u)du.$$

Properties of $\hat{G}(\cdot)$:

- $\hat{G}(\cdot)$ is a zero mean, continuous path Gaussian process.
- $\hat{G}(\cdot)$ is self-similar satisfying

$$\hat{G}(c\cdot) =^d \ c^{(3-\beta)/2}\hat{G}(\cdot), \quad c > 0.$$

- $\hat{G}(\cdot)$ has covariance function

$$\hat{C}(s,t) = \int_{u=0}^{s}\int_{v=0}^{t} C(u,v)dudv, \quad 0 \le s \le t.$$

3.2.3. Approximation for $X(\cdot)$.

The only way to get a **limit law** for $X(T)$ is to allow work rate r to depend on T; same philosophy as heavy traffic limits.

Fix $y > 0$, and set

$$r = r_T = m(Ty).$$

Let

$$X_T(t) = \text{content at } Tt \text{ when work rate}{=}r_T.$$

Then for $t < y$,

$$X_T(t) \Rightarrow 0$$

and in $D(y, \infty)$

$$\sqrt{m(T)}\Big(\frac{X_T(\cdot)}{Tm(T)} - c_T(\cdot)\Big) \Rightarrow \hat{G}(\cdot) - \hat{G}(y)$$

where $c_T(\cdot)$ is given

$$c_T(t) = \left(\frac{\int_{Ty}^{Tt} m(u)du}{Tm(T)}\right) - \left(\frac{m(Ty)}{m(T)}(t-y)\right).$$

Note $\hat{G}(\cdot)$ given in approximation for $A(\cdot)$.

3.3. Tidbit of disscussion

Recall

$$\text{input rate } = M(T) \overset{ip}{\sim} m(T)$$

so to have any hope of balance, need output rate of order $m(T)$. Write

$$X_T(t) = \bigvee_{s=0}^{Tt} \Big(A(Tt) - A(s) - r_T(Tt - s) \Big)$$

$$= \bigvee_{s=0}^{t} \Big(A(Tt) - A(Ts) - r_T T(t - s) \Big)$$

Now proceed:

1. Re-write $A(Tt)$ in terms of $\hat{G}_T(t)$. Do algebra.
2. To evaluate the sup of the process, take the derivative.
3. Try to set derivative =0. OK, you can't really do this. Look for turning point. Dream about what the answer would be if M replaced by its approximation.
4. Get properties of points where local maxima achieved.

NB: Can neaten things up if assume additionally *2nd order regular variation*.

4. Sessions initiated at renewal times. (Mikosch)

4.1. The model.

$\{S_n, n \geq 1\}$ = times of session initiations; ordinary renewal process;

$$S_0 = 0,\ S_n = \sum_{i=1}^{n} X_i;\ \{X_n\} \text{ iid};$$

$$X_i \sim F(x);\ \bar{F}(x) = 1 - F(x) \in RV_{-\alpha}$$

$\{T_n\}$ = session durations; iid non-negative rv's with common distribution $G(x)$ satisfying

$$\bar{G}(x) \in RV_{-\beta},$$

$$\{T_n\} \text{ independent of } \{S_n\}.$$

$$M(t) = \text{number of active sessions at } t, = \sum_{n=1}^{\infty} 1_{[S_n \leq t < S_n + T_n]}$$

$$A(t) = \int_0^t M(s)ds,$$

= cumulative work inputted in $[0,t]$, assuming unit rate input.

4.2. Cases.

1. Comparable tails: $\beta = \alpha$ and $\bar{F}(x) \sim c\,\bar{G}(x)$, $c > 0$, as $x \to \infty$.

    (a) The distribution tails of X_1 and T_1 are essentially the same.

    (b) For simplicity, we assume $c = 1$.

    (c) Kind of stability for M which converges weakly w/o normalization.

2. G heavier-tailed:

    (a) $0 < \beta < \alpha < 1$ or, if $\beta = \alpha$, then $\bar{F}(x)/\bar{G}(x) \to 0$ as $x \to \infty$ so that the distribution tail of X_1 is lighter than the distribution tail of T_1.

    (b) $0 = \beta < \alpha < 1$ so that the distribution tail of T_1 is slowly varying and thus again heavier than that of X_1.

    (a) Implies buildup in the M process.

3. F heavier-tailed: $\beta > \alpha$ so that the distribution tail of X_1 is heavier than the distribution tail of T_1.

    - Renewal epochs sparse relative to session lengths.
    - Of less interest.

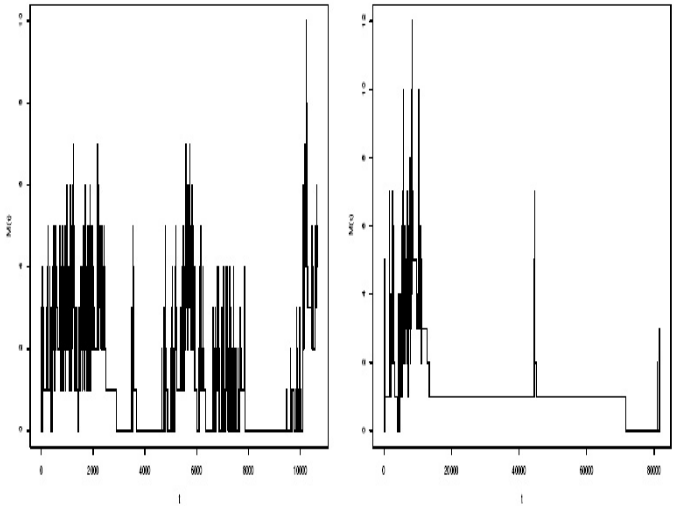

Figure 2: Paths of M; $\alpha = \beta = 0.9$ (left); $\alpha = \beta = 0.6$ (right).

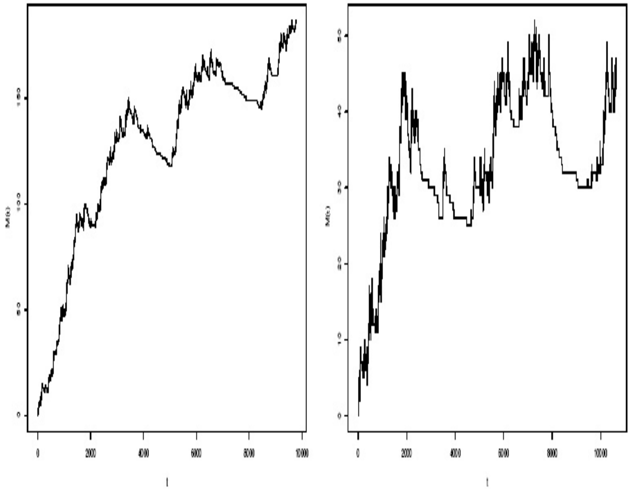

Figure 3: Paths of M; $(\alpha, \beta) = (0.9, 0.2)$ (left); $(\alpha, \beta) = (0.9, 0.4)$ (right).

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4.3. Warm-up: Mean value analysis for $\alpha, \beta < 1$.

Obtain asymptotic behavior of $E\big(M(t)\big)$ from Karamata's Tauberian theorem. Let

$$U(x) = \sum_{n=0}^{\infty} F^{n*}(x), \quad x > 0, \quad = \text{renewal function.}$$

Since $0 < \alpha < 1$, well known (eg Feller, 1971); as $x \to \infty$,

$$U(x) \sim \big(\Gamma(1-\alpha)\,\Gamma(1+\alpha)\,\bar{F}(x)\big)^{-1} \sim c(\alpha)\,x^{\alpha}/L_F(x).$$

Therefore ($t \to \infty$),

$$\mathbb{E}M(t) = \int_0^t U(dx)\,\bar{G}(t-x) = \int_0^1 \frac{\bar{G}(t(1-s))}{\bar{G}(t)}\frac{U(tds)}{U(t)}\Big(\bar{G}(t)U(t)\Big)$$

$$\sim c(\alpha)\int_0^1 (1-s)^{-\beta}\alpha s^{\alpha-1}ds\,\frac{\bar{G}(t)}{\bar{F}(t)} = c'(\alpha)\frac{\bar{G}(t)}{\bar{F}(t)}.$$

$$\mathbb{E}M(t) \sim c'(\alpha)\frac{\bar{G}(t)}{\bar{F}(t)}, \quad (t \to \infty).$$

Conclusions

- Case (1): Comparable tails.

  $$E\big(M(t)\big) \text{ converges to a constant.}$$

- Case (2): G is more heavy-tailed than F.

  $$\mathbb{E}M(t) \to \infty.$$

- Case (3): F is more heavy-tailed than G.

  $$\mathbb{E}M(t) \to 0.$$

  and hence

  $$M(t) \stackrel{L_1}{\to} 0.$$

  so Case (3) may be of lesser interest. (Renewals are sparse relative to event durations that at any time there is not likely to be an event in progress.)

4.4. Summary of results.

Table 1: Limiting behavior of $M(t)$ as $t \to \infty$.

Conditions	Limit behavior of $M(t)$ as $t \to \infty$
$0 < \alpha < 1$ $\bar{F} \sim \bar{G}$	$M(t) \Rightarrow$ random limit.
$0 \leq \beta < \alpha < 1$ or $0 < \alpha = \beta < 1$ and $\bar{F} = o(\bar{G})$	$\frac{\bar{F}(t)}{\bar{G}(t)} M(t) \Rightarrow$ random limit.
$0 < \beta < 1$ $\mathbb{E}(X_1) < \infty$	$\frac{M(t)}{t\bar{G}(t)} \Rightarrow$ constant $\frac{M(t) - \text{ random centering}}{\sqrt{t\bar{G}(t)}} \Rightarrow$ Gaussian rv
$0 < \beta \leq \alpha = 1$ $\mathbb{E}(X_1) = \infty$	$\frac{M(t)}{t\bar{G}(t)\mu(t)} \Rightarrow$ constant $\mu(t)=$ truncated 1st moment
$\mathbb{E}(X_1) < \infty$ $\mathbb{E}(T_1) < \infty$	Stationary version of $M(\cdot)$ exists

Focus on the first 2 rows corresponding to $\alpha,\ \beta < 1$.

4.5. Renewal: $\alpha, \beta < 1$, comparable tails.

A kind of stability exists for this case since fidi's of $M(t\cdot)$ converge in distribution to a limit.

4.5.1. Preliminaries

Define

$$N(x) = \sum_{n=0}^{\infty} 1_{[S_n \le x]} = \inf\{n : S_n > x\} = S^{\leftarrow}(x), \quad x \ge 0.$$

$$= \text{renewal counting function.}$$

$$\sum_k \epsilon_{(t_k, j_k)} = N_\infty = \text{ PRM(Leb} \times \nu_\alpha) \text{ on } [0,\infty) \times (0,\infty] := \mathbb{E}$$

$$\nu_\alpha(x,\infty] = x^{-\alpha}$$

$$X_\alpha(t) = \sum_{t_k \le t} j_k, \quad t \ge 0,$$

$$= \text{non-decreasing } \alpha\text{-stable Levy motion with Levy measure } \nu_\alpha.$$

$$b(t) \sim \left(\frac{1}{1-F}\right)^{\leftarrow}(t), \quad t \to \infty, \quad t\bar{F}(b(t)) \sim 1;$$

$$= \text{quantile function of } F$$

$$X^{(s)}(t) = \frac{S_{[st]}}{b(s)} \Rightarrow X_\alpha(t), \quad (s \to \infty),$$

$$= \text{renewal epochs are asymptotically stable.}$$

$$(X^{(s)})^{\leftarrow} \Rightarrow X_\alpha^{\leftarrow}, \quad \bar{F}(s)N(s\cdot) \Rightarrow X_\alpha^{\leftarrow}(\cdot)$$

$$\frac{1}{s}\sum_{n=0}^{\infty} \epsilon_{\frac{S_n}{b(s)}} \Rightarrow X_\alpha^{\leftarrow} \quad \text{in } M_+[0,\infty).$$

4.6. Comparable tails; $\alpha,\ \beta < 1$.

Define the time change map by

$$T : \mathbb{D}^{\uparrow}[0,\infty) \times M_+(\mathbb{E}) \mapsto M_+\big(\mathbb{E})$$

by

$$T(x,m) = \tilde{m}$$

where $\tilde{m}$ is defined by

$$\tilde{m}(f) = \iint f(x(u),v)\, m(du,dv)\,, \quad f \in C_K^+(\mathbb{E})\,.$$

If m is a point measure with representation $m = \sum_k \epsilon_{(\tau_k, y_k)}$, then

$$T(x,m) = \sum_k \epsilon_{(x(\tau_k),y_k)}\,.$$

Steps for analysis

1. in $M_p(\mathbb{E})$, as $s \to \infty$, regular variation of $\bar{G}$ equiv to

$$\sum_{k=0}^{\infty} \epsilon_{(\frac{k}{s}, \frac{T_k}{b(s)})} \Rightarrow N_\infty = PRM(\text{Leb} \times \nu_\alpha).$$

2. Since $\{S_k\}$ is independent of $\{T_k\}$, get joint convergence in $D[0,\infty) \times M_p(\mathbb{E})$,

$$\left(\frac{S_{[s\cdot]}}{b(s)}, \sum_{k=0}^{\infty} \epsilon_{(\frac{k}{s}, \frac{T_k}{b(s)})}\right) \Rightarrow (X_\alpha, N_\infty)\,.$$

3. Apply the a.s. continuous function T:

$$T\left(\frac{S[s\cdot]}{b(s)}, \sum_{k=0}^{\infty} \epsilon_{(\frac{k}{s}, \frac{T_k}{b(s)})}\right) = \sum_{k=0}^{\infty} \epsilon_{(\frac{S_{[sk/s]}}{b(s)}, \frac{T_k}{b(s)})} = \sum_{k=0}^{\infty} \epsilon_{(\frac{S_k}{b(s)}, \frac{T_k}{b(s)})} \Rightarrow T(X_\alpha, N_\infty)\,.$$

4. Evaluate on $\{(u,v) : u \le t \le u+v\}$ to get result for M: The fidi's of $M(t)$ satisfy as $s \to \infty$,

$$M(s\,t) = \sum_{k=0}^{\infty} 1_{[\frac{S_k}{s} \le t < \frac{S_k + T_k}{s}]} \Rightarrow M_\infty(t) = \sum_k 1_{[X_\alpha(t_k) \le t < X_\alpha(t_k) + j_k]}\,.$$

4.7. Case 2: $\bar{G}$ heavier; $\alpha, \beta < 1$.

Ingredients for analysis:

1. Recall $b(t)$ is the quantile function of F and satisfies

$$s\bar{F}(b(s)) \to 1, \quad (s \to \infty).$$

2. Since $\bar{G} \in RV_{-\beta}$,

$$\frac{s\,\bar{F}(b(s))}{\bar{G}(b(s))} G(b(s)\cdot) \xrightarrow{v} \nu_\beta.$$

in $M_+(0,\infty]$, where $\xrightarrow{v}$ denotes vague convergence.

3. Equivalent to previous convergence is

$$\frac{\bar{F}(b(s))}{\bar{G}(b(s))} \sum_{k=0}^{[s]} \varepsilon_{\frac{T_k}{b(s)}} \Rightarrow \nu_\beta\,.$$

4. Extend by adding time component:

$$\frac{\bar{F}(b(s))}{\bar{G}(b(s)} \sum_{k=0}^{\infty} \varepsilon_{(\frac{k}{s},\frac{T_k}{b(s)})} \Rightarrow \text{Leb} \times \nu_\beta\,.$$

5. Augment using independence of $\{T_n\}$ and $\{S_n\}$:

$$\left(\frac{S_{[s\cdot]}}{b(s)}, \frac{\bar{F}(b(s))}{\bar{G}(b(s))} \sum_{k=0}^{\infty} \varepsilon_{\left(\frac{k}{s}, \frac{T_k}{b(s)}\right)}\right) \Rightarrow (X_\alpha, \text{Leb} \times \nu_\beta) .$$

6. Apply the a.s. continuous map T; evaluate the result on the correct region to get result for M: The fidi' s of M satisfy

$$\frac{\bar{F}(s)}{\bar{G}(s)} M(s\,t) \Rightarrow \int_0^t (t-u)^{-\beta}\, dX_\alpha^{\leftarrow}(u). \quad s \to \infty.$$

5. Cumulative work process.

Sample results for workload process

$$A(t) = \int_0^t M(s)ds.$$

5.1. The case $\mu_X < \infty$, $\beta \in (1,2)$

Define the quantile function of G:

$$\sigma(t) \sim \left(\frac{1}{1-G}\right)^{\leftarrow}(t)\,, \quad t \to \infty.$$

Assume either $\bar{F} \in RV_{-\alpha}, 1 < \alpha \le 2$ or $\sigma_X^2 < \infty$.

1. Suppose $\bar{F} \in RV_{-\alpha}$ and either
   (a) $\alpha > \beta$ or
   (b) $\alpha = \beta$ and $\bar{F}(x) = o(\bar{G}(x))$ or
   (c) $\sigma_X^2 < \infty$.

   Set
   $$A_s(u) = \sigma(s)^{-1}\Big(A(su) - su\mu_T/\mu_X\Big), \quad u \ge 0.$$
   Then $(s \to \infty)$,
   $$A_s(\cdot) \Rightarrow \mu_X^{-1/\beta} X_\beta(\cdot)\,, \tag{1}$$

where X_β is a β-stable spectrally positive Lévy motion on $[0,\infty)$.

2. If $\bar F \in RV_{-\alpha}$, $\alpha = \beta$ and $\bar F(x) \sim c\,\bar G(x)$, then (1) holds, where X_β is β-stable Lévy motion with a skewness parameter.

3. If $\bar F \in RV_{-\alpha}$ and $\alpha < \beta$ or $\alpha = \beta$ and $\bar G(x) = o(\bar F(x))$, then, as $s \to \infty$

$$(b(s))^{-1}\left[A(\cdot\, s) - s\,(\cdot)\,\mu_T/\mu_X\right] \Rightarrow \mu_X^{-1/\alpha}\, X_\alpha(\cdot)\,,$$

where X_α is spectrally negative α-stable Lévy motion.

5.2. $\alpha, \beta < 1$.

Case (2) assumptions hold: $0 \le \beta \le \alpha < 1$ and if $\alpha = \beta$, then $\bar F(s)/\bar G(s) \to 0$, as $s \to \infty$. Then

$$\frac{\bar F(s)}{s\bar G(s)} A(st) \Rightarrow \int_0^t \frac{(t-u)^{1-\beta}}{1-\beta} dX_\alpha^{\leftarrow}(u), \quad t \ge 0\,,$$

in $C[0,\infty)$.

NB: This is the integrated version of the result for M.