Multivariate Heavy Tails, Asymptotic Independence and Beyond

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Work with: K. Maulik, J. Heffernan, S. Marron, ...

1. Multidimensional Heavy Tails.

Consider a vector $\boldsymbol{X} = (X^{(1)}, \ldots, X^{(d)})$ where

- The components may be dependent.
- The components are each univariate heavy tailed.

Big issue: How to model the dependence?

- The tail indices (α's) for each component are typically different in practice.
- Parametric (use MLE) vs semi-parametric (use asymptotic theory).
  - Parametric will fail goodness of fit with large data sets.
  - Semi-parametric will have difficult asyptotic theory.
- Stable and max-stable distributions indexed by measures on the unit sphere–large classes and why should even the marginals be correct? Parametric sub-families may be ad hoc.
- Copula methods.

1.1. Example.

Internet traffic:
Consider

$$\begin{aligned} F &= \text{file size}, \\ L &= \text{duration of transmission}, \\ R &= \text{throughput } = F/L. \end{aligned}$$

All three, are seen empirically to be heavy tailed.

Two studies:

- BU
- UNC

What is the dependence structure of (F, R, L)?
Since $F = LR$, the tail parameters $(\alpha_F, \alpha_R, \alpha_L)$ cannot be arbitrary.

Note for BU measurements, we have the following empirical estimates:

α	$\hat\alpha_F$	$\hat\alpha_R$	$\hat\alpha_L$
estimated value	1.15	1.13	1.4

Two theoretical possibilities:

- If (L, R) have a joint distribution with multivariate regularly varying tail but are NOT asymptotically independent then (Maulik, Resnick, Rootzen (2002))

$$\hat\alpha_F = \frac{\hat\alpha_L \hat\alpha_R}{\hat\alpha_L + \hat\alpha_R} = .625 \neq 1.15.$$

- If (L, R) obey a form (not the EVT version) of asymptotic independence, (Maulik+Resnick+Rootzen; Heffernan+Resnick)

$$tP[\Big(L, \frac{R}{b(t)}\Big) \in \cdot\,] \xrightarrow{v} G \times \alpha x^{-\alpha-1} dx$$

then

$$\alpha_F = \alpha_R \bigwedge \alpha_L$$

and in our example

$$1.15 \approx 1.13 \bigwedge 1.4.$$

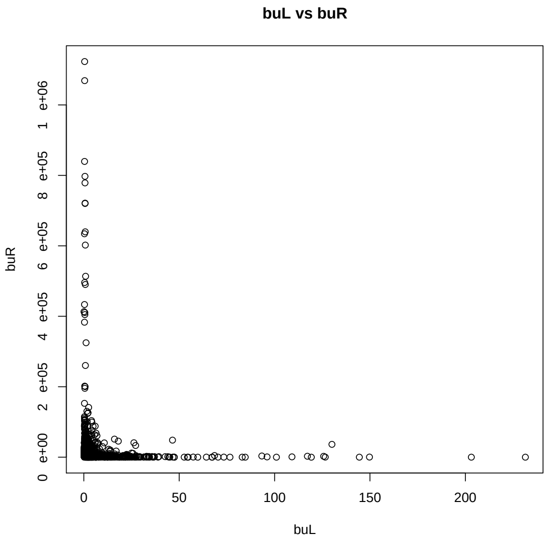
For two examples

- BU: Evidence seems to support some form of independence for (R, L).
- UNC: Conclusions from Campos, Marron, Resnick, Jeffay (2005);
  - Large values of F tend to be independent of large values of R.

    $\Rightarrow$ Large files do not seem to receive any special consideration when rates are assigned.

BuL vs BuR:

Data processed from the original 1995 Boston University data; 4161 file sizes (F) and download times (L) noted and transmission rates (R) inferred. The data consists of bivariate pairs (R,L).



2. Multivariate Regular Variation.

2.1. Standard Case

A fct $U : \mathbb{R}_+^d \mapsto \mathbb{R}_+$ is mult reg varying if

$$\frac{U(t\boldsymbol{x})}{U(t\mathbf{1})} \to \lambda(\boldsymbol{x}) \neq 0,$$

for $\boldsymbol{x} \geq \mathbf{0}$, $\boldsymbol{x} \neq \mathbf{0}$. Then $\exists$ ρ and

$$\lambda(t\boldsymbol{x}) = t^\rho \lambda(\boldsymbol{x}),$$

and $U(t\mathbf{1}) \in RV_\rho$.

Usually there is a sequential equivalent version: $\exists$ $b_n \to \infty$ such that

$$\frac{U(b_n\boldsymbol{x})}{n} \to \lambda(\boldsymbol{x}).$$

Application to distributions: For simplicity, let $\boldsymbol{Z}, \boldsymbol{Z}_n, n \geq 1$ be iid, range=$\mathbb{R}_+^d$ and common df F. A *regularly varying tail* means

$$\frac{1 - F(t\boldsymbol{x})}{1 - F(t\mathbf{1})} \to \nu([\mathbf{0}, \boldsymbol{x}]^c),$$

for some Radon measure ν. However, it is awkward to deal with mult df's and better to deal with measures.

Let

$$\begin{aligned} \mathbb{E} =& [0, \infty]^d \setminus \{\mathbf{0}\} \\ \aleph =& \{\boldsymbol{x} \in \mathbb{E} : \|\boldsymbol{x}\| = 1\}, \\ R =& \|\boldsymbol{Z}\|, \quad \boldsymbol{\Theta} = \frac{\boldsymbol{Z}}{\|\boldsymbol{Z}\|} \in \aleph. \end{aligned}$$

The following are equivalent and define multivariate heavy tails or regularly varying tails.

1. $\exists$ a Radon measure ν on $\mathbb{E}$ such that

$$\lim_{t\to\infty}\frac{1-F(t\boldsymbol{x})}{1-F(t\mathbf{1})}=\lim_{t\to\infty}\frac{\mathbb{P}\big[\frac{\boldsymbol{Z}_1}{t}\in[\mathbf{0},\boldsymbol{x}]^c\big]}{\mathbb{P}\big[\frac{\boldsymbol{Z}_1}{t}\in[\mathbf{0},\mathbf{1}]^c\big]}$$

$$=c\nu\Big([\mathbf{0},\boldsymbol{x}]^c\Big),$$

some $c>0$ and for all points $\boldsymbol{x}\in[\mathbf{0},\boldsymbol{\infty})\setminus\{\mathbf{0}\}$ which are continuity points of $\nu([\mathbf{0},\cdot]^c)$.

2. $\exists$ a function $b(t)\to\infty$ and a Radon measure ν on $\mathbb{E}$ such that in $M_+(\mathbb{E})$

$$t\mathbb{P}\Big[\frac{\boldsymbol{Z}_1}{b(t)}\in\cdot\Big]\xrightarrow{v}\nu,\quad t\to\infty.$$

3. $\exists$ a pm $S(\cdot)$ on $\aleph$ and $b(t)\to\infty$ such that

$$t\mathbb{P}[\big(\frac{R_1}{b(t)},\boldsymbol{\Theta}_1\big)\in\cdot]\xrightarrow{v}c\nu_\alpha\times S$$

in $M_+\Big(\big((0,\infty]\times\aleph\big)\Big)$, where $c>0$ and

$$\nu_\alpha(x,\infty]=x^{-\alpha}.$$

4. $\exists$ $b_n\to\infty$ such that in $M_p(\mathbb{E})$

$$\sum_{i=1}^n\epsilon_{\boldsymbol{Z}_i/b_n}\Rightarrow \mathrm{PRM}(\nu).$$

5. $\exists$ a sequence $b_n \to \infty$ such that in $M_p((0,\infty] \times \aleph)$

$$\sum_{i=1}^{n} \epsilon_{(R_i/b_n,\Theta_i)} \Rightarrow \text{PRM}(c\nu_\alpha \times S).$$

These conditions imply that for any sequence $k = k(n) \to \infty$ such that $n/k \to \infty$ we have

6. In $M_+(\mathbb{E})$,

$$\frac{1}{k}\sum_{i=1}^{n} \epsilon_{\boldsymbol{Z}_i/b(\frac{n}{k})} \Rightarrow \nu \qquad (*)$$

$$\frac{1}{k}\sum_{i=1}^{n} \epsilon_{(R_i/b(n/k),\Theta_i)} \Rightarrow (c\nu_\alpha \times S). \qquad (**)$$

and (6) is equivalent to any of (1)–(5), provided $k(\cdot)$ satisfies $k(n) \sim k(n+1)$.

Ignore fact $b(\cdot)$ unknown:

$\to$ LHS of Eqn (*) is a consistent estimator of ν.

$\to$ From (**), consistent estimator of S is

$$\frac{\sum_{i=1}^{n} \epsilon_{(R_i/b(n/k),\Theta_i)}[1,\infty] \times \cdot)}{\sum_{i=1}^{n} \epsilon_{R_i/b(n/k)}[1,\infty]}.$$

But:

- This theoretical formulation is for the *standard case*.
  - Problematic for applications. If we norm each component by the same $b(t) \Rightarrow$ marginal tails same; ie components on the same scale:

$$\mathbb{P}[Z^{(i)} > x] \sim c_{ij}\mathbb{P}[Z^{(j)} > x], \quad c_{ij} > 0,\ x \to \infty.$$

  - Standard case almost never happens in practice.
- How to transform to the standard case in practice?
  - Simple minded: Hope $1 - F_{(i)}(x) \sim x^{-\alpha_i}$ for all i and then power up. BUT: Must estimate α's. YECH!
  - Use ranks method (Huang, 1992; de Haan & de Ronde). BUT: Lose independence among observations.

The ranks method:

Given d-dimensional random vectors $\{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n\}$ where

$$\boldsymbol{X}_i = (X_i^{(1)}, \ldots, X_i^{(d)}), \quad i = 1, \ldots, n,$$

define the (anti)-ranks for each component: Comparing the jth components, $X_1^{(j)}, \ldots, X_n^{(j)}$, the anti-rank of $X_i^{(j)}$ is

$$r_i^{(j)} = \sum_{l=1}^{n} 1_{[X_l^{(j)} \geq X_i^{(j)}]}$$

$$= \#\text{ jth components} \geq X_i^{(j)}.$$

Replace each $\boldsymbol{X}_i$ by

$$\boldsymbol{X}_i \mapsto \big(1/r_i^{(j)},\ j = 1, \ldots, d\big).$$

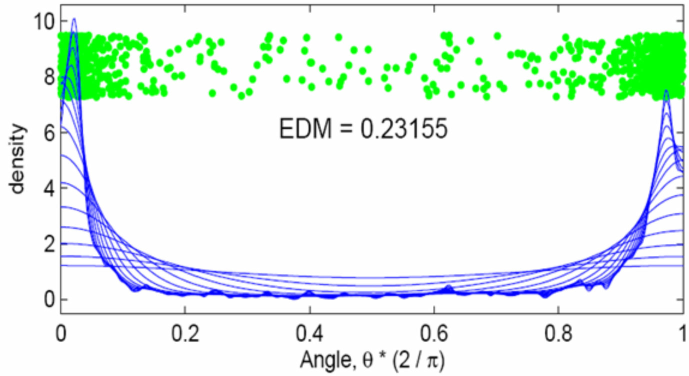
Rank method–UNC

Steps:

- Transform (F,R) data using rank method.
- Convert to polar coordinates.
- Keep 2000 pairs with biggest radius vector.
- Compute density estimate for angular measure S.

Plot: Density estimates with various amounts of smoothing+jitter plot (green) of angles.

Full disclosure: These types of plots can be rather sensitive to choice of threshold.



EDM = 0.23155
density
Angle, θ * (2 / π)
0
2
4
6
8
10
0
0.2
0.4
0.6
0.8
1

2.2. Simplifying assumptions

For theory, proceed assuming

- Standard case.
- One dimensional marginals $F_{(i)}$, $i = 1, \dots, d$ are the same.
- $d = 2$ (just for ease of explanation).

3. Significance of limit measure

The limit measure ν controls the (asymptotic) dependence structure: The distribution F of $\boldsymbol{Z}_1$ possesses *asymptotic independence* if either

1. $\nu\big((\mathbf{0},\boldsymbol{\infty})\big)=0$ so that ν concentrates on the axes;

OR

2. S concentrates on $\{(1,0),(0,1)\}$.

This definition designed to yield

- As $n\to\infty$
$$\bigvee_{i=1}^{n}\frac{\boldsymbol{Z}_i}{b_n}\Rightarrow(Y^{(1)},Y^{(2)}),$$
where $(Y^{(1)},Y^{(2)})$, are independent Frechet distributed.
- Probability of 2 components being simultaneously large is negligible: For $d=2$:
$$\lim_{t\to\infty}\mathbb{P}[Z^{(2)}>t|Z^{(1)}>t]\to 0.$$

3.1. Why asymptotic independence creates problems.

- Estimators of various parameters may behave badly under asymptotic independence; eg, estimator of the spectral measure S. Estimators may be asymptotically normal with an asymptotic variance of 0 (oops!).
- Estimators of probabilities given by asymptotic theory may be uninformative.

Scenario: Estimate the probability of simultaneous non-compliance.

Supppose $\boldsymbol{Z} = (Z^{(1)}, Z^{(2)})$ = concentrations of different pollutants. Environmental agencies set critical levels $\boldsymbol{t}_0 = (t_0^{(1)}, t_0^{(2)})$ which not be exceeded. Imagine simultaneous *non-compliance* creates a health hazard. Worry about

$$[\text{ health hazard }] = [\boldsymbol{Z} > \boldsymbol{t}_0] = [Z^{(j)} > t_0^{(j)};\ j = 1, 2].$$

Assume only regular variation with unequal components. Then for the probability of *non-compliance*, we estimate

$$\begin{aligned} P[Z^{(1)} > t_0^{(1)}, Z^{(2)} > t_0^{(2)}] =& P[\frac{Z^{(j)}}{b^{(j)}(\frac{n}{k})} > \frac{t_0^{(j)}}{b^{(j)}(\frac{n}{k})}; j = 1, 2] \\ \approx& \frac{k}{n}\nu\Bigg(\Bigg(\Big(\frac{t_0^{(1)}}{b^{(1)}(\frac{n}{k})}, \frac{t_0^{(2)}}{b^{(2)}(\frac{n}{k})}\Big), \boldsymbol{\infty}\Bigg]\Bigg) = \quad 0 \end{aligned}$$

since ν has empty interior by asymtotic independence.

This is not helpful!!

4. Hidden Regular Variation.

A submodel of asymptotic independence.

The random vector $\boldsymbol{Z}$ has a distribution possessing *hidden regular variation* if

1. Regular variation on the big cone $\mathbb{E} = [0,\infty]^2 \setminus \{\mathbf{0}\}$:

$$t\mathbb{P}[\frac{\boldsymbol{Z}}{b(t)} \in \cdot] \xrightarrow{v} \nu,$$

AND

2. Regular variation on the small cone $(0,\infty]^2$: $\exists$ a non-decreasing function $b^*(t) \uparrow \infty$ such that

$$b(t)/b^*(t) \to \infty$$

and $\exists$ a measure $\nu^* \neq 0$ which is Radon on $\mathbb{E}^0 = (0,\infty]^2$ and such that

$$tP[\frac{\boldsymbol{Z}}{b^*(t)} \in \cdot] \xrightarrow{v} \nu^* \;=\; \text{hidden measure}$$

on the cone $\mathbb{E}^0$.

Then there exists $\alpha^* \geq \alpha$ such that $b^* \in RV_{1/\alpha^*}$.

Consequences:

- With the right formulation,

$$\begin{aligned}&\text{Second order regular variation + asy indep}\\ &\quad\Rightarrow \text{hidden regular variation}\\ &\quad\Rightarrow \text{asymptotic independence.}\end{aligned}$$

- Means for every $\boldsymbol{s} \geq 0$, $\boldsymbol{s} \neq 0$, $\bigvee_{i=1}^{d} s^{(i)} Z^{(i)}$ has distribution with a regularly varying tail of index α and for every $\boldsymbol{a} \geq \mathbf{0}, \boldsymbol{a} \neq \mathbf{0}$, $\bigwedge_{i=1}^{d} a^{(i)} Z^{(i)}$ has a regularly varying distribution tail of index α^*.

- In particular, hidden regular variation means both $Z^{(1)} \vee Z^{(2)}$ and $Z^{(1)} \wedge Z^{(2)}$ have regularly varying tail probabilities with indices α and α^*. Note

$$\eta = 1/\alpha_* = \text{coefficient of tail dependence}$$

(Ledford and Tawn (1996,1997)).

- Define on $\aleph \cap \mathbb{E}^0$

$$S^*(\Lambda) = \nu^*\{\boldsymbol{x} \in \mathbb{E}^0 : |\boldsymbol{x}| \geq 1, \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \in \Lambda\}$$

called the *hidden angular measure.*

Sub-model (cont)–Two Examples:

Example 1: $d = 2$; independent random quantities $B, \boldsymbol{Y}, \boldsymbol{U}$ with

$$P[B = 0] = P[B = 1] = 1/2$$

and $\boldsymbol{Y} = (Y^{(1)}, Y^{(1)})$ is iid with

$$P[Y^{(1)} > x] \in RV_{-1}$$

and

$$b(t) = \Big(\frac{1}{P[Y^{(1)} > \cdot]}\Big)^{\leftarrow}(t) \in RV_1.$$

Let $\boldsymbol{U}$ have multivariate regularly varying distribution on $\mathbb{E}$ and $\exists \alpha^* > 1$, $b^*(t) \in RV_{1/\alpha^*}$, $\nu^* \not\equiv 0$,

$$tP[\frac{\boldsymbol{U}}{b^*(t)} \in \cdot] \to \nu^* \neq 0.$$

Define

$$\boldsymbol{Z} = B\boldsymbol{Y} + (1 - B)\boldsymbol{U}$$

which has hidden regular variation, and the property

$$\boxed{S^*(\aleph^0) := \nu^*\{\boldsymbol{x} \in \mathbb{E}^0 : \|\boldsymbol{x}\| > 1\} < \infty.}$$

Example 2: $d = 2$, define

$$\nu^*\big([\boldsymbol{x}, \boldsymbol{\infty}]\big) = \big(x^{(1)}x^{(2)}\big)^{-1}.$$

Define $\boldsymbol{Z} = (Z^{(1)}, Z^{(2)})$ iid and Pareto distributed with

$$P[Z^{(1)} > x] = x^{-1}, \quad x > 1,\ i = 1, 2.$$

Set

$$b(t) = t, \quad b^*(t) = \sqrt{t},$$

so that $b(t)/b^*(t) \to \infty$. Then on $\mathbb{E}$

$$tP[\frac{\boldsymbol{Z}}{b(t)} \in \cdot] \to^v \nu,$$

$\nu(\mathbb{E}^0) = 0$, and on $\mathbb{E}^0$

$$tP[\frac{\boldsymbol{Z}}{b^*(t)} \in \cdot] \to^v \nu^*,$$

and

$$\boxed{S^*(\aleph^0) := \nu^*\{\boldsymbol{x} \in \mathbb{E}^0 : \|\boldsymbol{x}\| > 1\} = \infty.}$$

How dense are these 2 examples?

Need for a concept of multivariate tail equivalence: Sppse

$$\mathbf{0} \le \boldsymbol{Y} \sim F; \quad \mathbf{0} \le \boldsymbol{Z} \sim G.$$

Say F, G (or $\boldsymbol{Y}$ and $\boldsymbol{Z}$) *are tail equivalent on cone* $\mathbb{C}$ if there exists $b(t) \uparrow \infty$ such that

$$tP\left[\boldsymbol{Y}/b(t) \in \cdot\right] = tF(b(t)\cdot) \xrightarrow{v} \nu$$

and

$$tP\left[\boldsymbol{Z}/b(t) \in \cdot\right] = tG(b(t)\cdot) \xrightarrow{v} c\nu$$

for $c > 0$, Radon $\nu \neq 0$ on $\mathbb{C}$.
Write

$$\boldsymbol{Y} \stackrel{te(\mathbb{C})}{\sim} \boldsymbol{Z}.$$

5. Characterizations.

Mixture Characterization; S^* is Finite

Assume finite hidden angular measure: Sppse $\boldsymbol{Z} \sim F$ is multivariate regularly varying on

$$\begin{array}{lll} \mathbb{E} := [\mathbf{0}, \boldsymbol{\infty}]^d \setminus \{\mathbf{0}\}, & \text{scaling } b(t), & \\ \mathbb{E}_0 := (\mathbf{0}, \boldsymbol{\infty}]^d, & \text{scaling } b^*(t), & b(t)/b^*(t) \to \infty, \\ b \in RV_{1/\alpha}, & b^* \in RV_{1/\alpha^*}, & \alpha \leq \alpha^*. \end{array}$$

Then F is tail equivalent on both the cones $\mathbb{E}$ and $\mathbb{E}_0$ to a mixture distribution

$$\boldsymbol{Z} \overset{te(\mathbb{C})}{\sim} 1_{[I=0]} \boldsymbol{V} + \sum_{i=1}^{d} 1_{[I=i]} X_i \boldsymbol{e}_i .$$

Here $\boldsymbol{e}_i; i = 1, \dots, d$ are the usual basis vectors.

Remarks on the characterization:

$$\boldsymbol{Z} \overset{te(\mathbb{C})}{\sim} 1_{[I=0]}\boldsymbol{V} + \sum_{i=1}^{d} 1_{[I=i]} X_i \boldsymbol{e}_i.$$

- $\sum_{i=1}^{d} 1_{[I=i]} X_i \boldsymbol{e}_i$ concentrates on the axes, has no hidden regular variation, and the marginal distributions (of the X_i) have scaling function $b(t)$,
- $\boldsymbol{V}$ mult reg varying on $\mathbb{E}$ (not $\mathbb{E}_0$–this is the effect of finite ν^*) with scaling function $b^*(t)$; tails of $\boldsymbol{V}$ are lighter than those of the completely asymptotically independent distribution $\sum_{i=1}^{d} 1_{[I=i]} X_i \boldsymbol{e}_i$.
- Conversely: if F tail equivalent to a mixture as above, $b(t)/b^*(t) \to \infty$, then F is multivariate reg varying on $\mathbb{E}$ and $\mathbb{E}_0$ with finite hidden angular measure and with scaling functions b, b^*.

Mixture Characterization; S^* is Infinite

Assume infinite hidden angular measure. Sppse $\boldsymbol{Z} \sim F$ mult regularly varying on

$$\begin{aligned} &\mathbb{E} := [\mathbf{0}, \boldsymbol{\infty}]^d \setminus \{\mathbf{0}\}, && \text{scaling } b(t), && \\ &\mathbb{E}_0 := (\mathbf{0}, \boldsymbol{\infty}]^d, && \text{scaling } b^*(t), && b(t)/b^*(t) \to \infty, \\ &b \in RV_{1/\alpha}, && b^* \in RV_{1/\alpha^*}, && \alpha \le \alpha^*. \end{aligned}$$

Then F is tail equivalent on both the cones $\mathbb{E}$ and $\mathbb{E}_0$ to a mixture distribution

$$\boldsymbol{Z} = 1_{[I=0]}\boldsymbol{V} + \sum_{i=1}^{d} 1_{[I=i]} X_i \boldsymbol{e}_i.$$

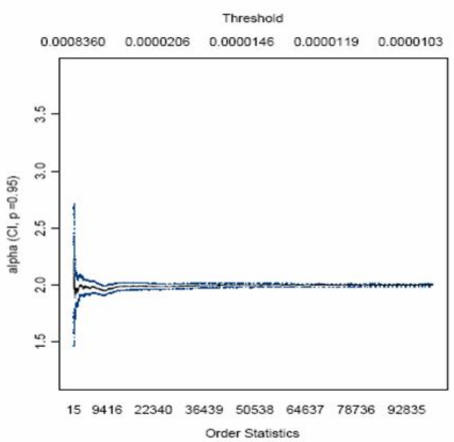
Remarks and notes on the infinite case:

- $\boldsymbol{V}$ is only guaranteed to be reg varying on $\mathbb{E}_0$; index is α^* .
- If the reg variation of $\boldsymbol{V}$ can be extended to $\mathbb{E}$, then the 1-dim marginals have heavier tails of index $\le \alpha^*$.
- BUT: do not have a useful criterion for when reg var on $\mathbb{E}_0$ can be extended to $\mathbb{E}$.

6. Can We Detect Hidden Regular Variation?

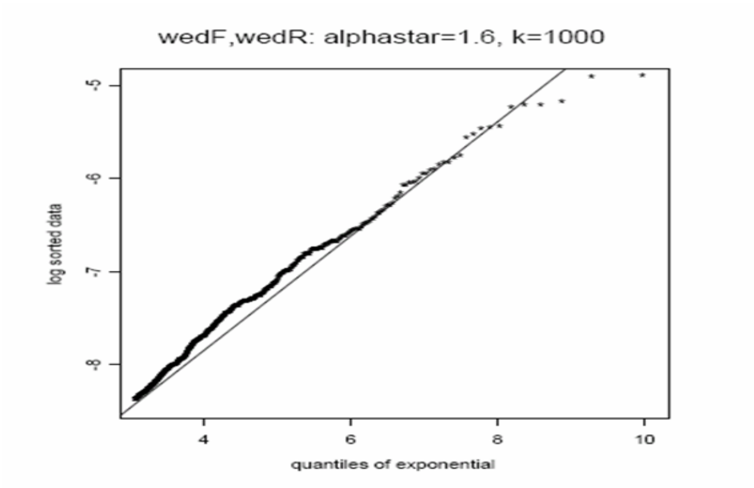
Example 1: Simulation.

5000 pairs of iid Pareto, $\alpha = 1$; $\alpha_* = 2$. Hillplot for rank transformed data taking minima of components.



Example 2: UNC Wed (F,R).

QQ plot of rank transformed data using 1000 upper order statistics for UNC Wed (F,R); $\alpha = 1$ and $\hat{\alpha}_* = 1.6$.



6.1. Estimating ν^*.

The hidden measure ν^* has a spectral measure S^* defined on $\aleph_0$, the unit sphere in $\mathbb{E}_0$:

$$S^*(\Lambda) := \nu^*\{\boldsymbol{x} \in \mathbb{E}_0 : \|\boldsymbol{x}\| > 1, \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \in \Lambda\}.$$

S^* may not necessarily be finite.

We estimate S^* rather than ν^*.

Estimation procedure (Heffernan & Resnick) for estimating ν^*:

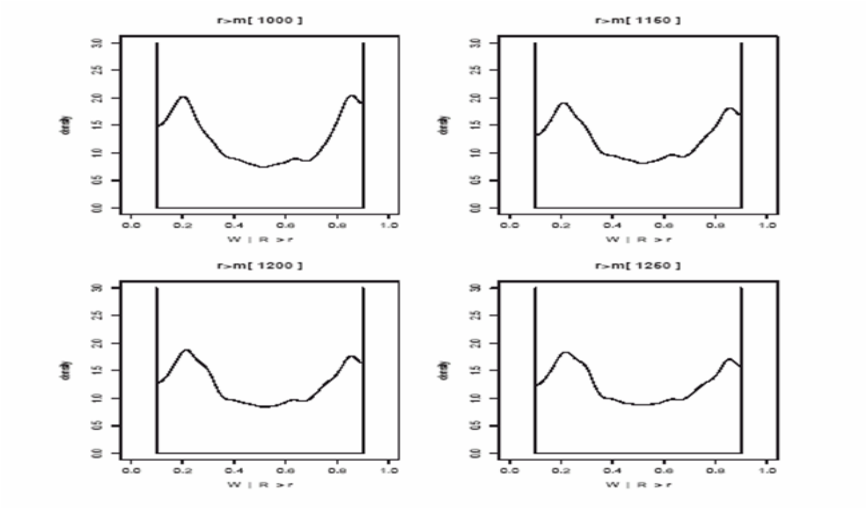
1. Replace the heavy tailed multivariate sample $\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_n$ by the n vectors of reciprocals of anti-ranks $1/\mathbf{r}_1, \dots, 1/\mathbf{r}_n$, where
$$r_i^{(j)} = \sum_{l=1}^{n} 1_{[Z_l^{(j)} \geq Z_i^{(j)}]}; \quad j = 1, \dots, d; \ i = 1, \dots, n.$$
2. Compute normalizing factors
$$m_i = \bigwedge_{j=1}^{d} \frac{1}{r_i^{(j)}}; \quad i = 1, \dots, n,$$
and their order statistics
$$m_{(1)} \geq \cdots \geq m_{(n)}.$$
3. Compute the polar coordinates $\{(R_i, \boldsymbol{\Theta}_i); i = 1, \dots, n\}$ of
$$\{(1/r_i^{(j)}; j = 1, \dots, d); \ i = 1, \dots, n\}.$$
4. Estimate S^* using the $\boldsymbol{\Theta}_i$ corresponding to $R_i \geq m_{(k)}$.

Details:

- If ν^* is infinite, let $\aleph_0(K)$ be compact subset of $\aleph_0$.
  - For $d = 2$ where $\aleph$ can be parameterized as $\aleph = [0, \pi/2]$ and $\aleph_0 = (0, \pi/2)$, set $\aleph_0(K) = [\delta, \pi/2 - \delta]$ for some small $\delta > 0$.
- Then
$$\frac{\sum_{i=1}^{n} 1_{[R_i \geq m_{(k)}, \boldsymbol{\Theta}_i \in \aleph_0(K)]} \epsilon_{\boldsymbol{\Theta}_i}}{\sum_{i=1}^{n} 1_{[R_i \geq m_{(k)}, \boldsymbol{\Theta}_i \in \aleph_0(K)]}} \Rightarrow S_0(\cdot \bigcap \aleph_0(K)).$$
- If ν^* is finite, we can replace $\aleph_0(K)$ with $\aleph_0$.

Example.

UNC (F,R), April 26. Asymptotic independence present. Since S^* may be infinite, we restricted estimation to the angular interval interval [0.1,0.9] instead of all of $[0, 1]$. All plots show the hidden measure to be bimodal with peaks around 0.2 and 0.85.



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7. Conditional models.

Other form of asymptotic independence (Maulik, Resnick, Rootzen):

$$nP\Big[\Big(X,\frac{Y}{b(n)}\Big)\in\cdot\Big]\xrightarrow{v} G\times\nu_\alpha \tag{1}$$

on $[0,\infty]\times(0,\infty]$ where G is a pm on $[0,\infty)$ and

$$\nu_\alpha(x,\infty]=x^{-\alpha},\ x>0.$$

Equivalent: Y has a regularly varying tail and

$$P[X\le x|Y>t]\xrightarrow{t\to\infty} G(x).$$

Heffernan & Tawn models:

$$P[\frac{X-\beta(t)}{\alpha(t)}\le x|Y=t]\xrightarrow{t\to\infty} G(x).$$

With Jan Heffernan: Meld 2 approaches. Reformulate as

$$tP\Big[\Big(\frac{X-\beta(t)}{\alpha(t)},\frac{Y-b(t)}{a(t)}\Big)\in\cdot\Big]\xrightarrow{v}\mu$$

where μ satisfies non-degeneracy assumptions.

7.1. Basic Convergence

Assume 2 dimensions and

$$tP\Big[\Big(\frac{X-\beta(t)}{\alpha(t)},\frac{Y-b(t)}{a(t)}\Big)\in\cdot\Big]\xrightarrow{v}\mu(\cdot),\qquad(2)$$

in $M_+\big([-\infty,\infty]\times(-\infty,\infty]\big)$, and non-degeneracy assumptions:

1. for each fixed y, $\mu\big((-\infty,x]\times(y,\infty]\big)$ is not a degenerate distribution function in x;
2. for each fixed x, $\mu\big((-\infty,x]\times(y,\infty]\big)$ is not a degenerate distribution function in y,

Observations:

- The Basic Convergence (2) implies

  $$tP\Big[\frac{Y-b(t)}{a(t)})\in\cdot\Big]\xrightarrow{v}\mu\big([-\infty,\infty]\times(\cdot)\big),$$

  so $P[Y\in\cdot]\in D(G_\gamma)$, for some $\gamma\in\mathbb{R}$.

- The Basic Convergence (2) implies the conditioned limit

  $$tP\Big[\frac{X-\beta(t)}{\alpha(t)}\le x|Y>b(t)\Big]\to\mu\big([-\infty,x]\times(0,\infty]\big).$$

- WLOG can assume Y is heavy tailed and reduce the basic convergence to standard form:

$$tP\Big[\Big(\frac{X-\beta(t)}{\alpha(t)}, \frac{Y}{t}\Big) \in \cdot\Big] \xrightarrow{v} \mu \tag{3}$$

  in $M_+([-\infty,\infty] \times (0,\infty])$ (with a modified μ).

- Suppose (X, Y) are regularly varying on $[0,\infty]^2 \setminus \{\mathbf{0}\}$.
  - With no asymptotic independence, Basic Convergence automatically holds.
  - With asymptotic independence, Basic Convergence is an extra assumption.

7.2. More reduction.

More remarks:

- A convergence to types argument implies variation properties of $\alpha(\cdot)$ and $\beta(\cdot)$: Suppose (X, Y) satisfy the standard form condition (3). $\exists$ two functions $\psi_1(\cdot), \psi_2(\cdot)$, such that for all $c > 0$,

$$\lim_{t\to\infty} \frac{\alpha(tc)}{\alpha(t)} = \psi_1(c), \quad \lim_{t\to\infty} \frac{\beta(tc) - \beta(t)}{\alpha(t)} \to \psi_2(c).$$

  locally uniformly.

- $\exists$ important cases where $\psi_2 \equiv 0$ (bivariate normal).

- Can sometimes also standardize the X variable so that

$$tP\Big[\frac{\beta^{\leftarrow}(X)}{t} \le x, \frac{Y}{t} > y\Big] \to \mu\big([-\infty, \psi_2(x)] \times (y, \infty]\big). \qquad (4)$$

When?? Short version: When μ is not a product measure.

  - $\mu = H \times \nu_1$ iff $\psi_1 \equiv 1$ ($\alpha(\cdot)$ is sv) and $\psi_2 \equiv 0$.
  - If $\beta(t) \ge 0$ and $\beta^{\leftarrow}$ is non-decreasing on the range of X, then (4) is possible iff μ is NOT a product.
  - A transformation of X allows one to bring the problem to the previous case.

- If we have $X \ge 0$ and both regular variation on $C_2 = [0, \infty]^2 \setminus \{\mathbf{0}\}$

$$tP\Big[\Big(\frac{X}{a'(t)}, \frac{Y}{t}\Big) \in \cdot\Big] \xrightarrow{v} \nu_*$$

and (4):

$$tP\Big[\frac{\beta^{\leftarrow}(X)}{t} \le x, \frac{Y}{t} > y\Big] \to \mu\big([-\infty, \psi_2(x)] \times (y, \infty]\big)$$

on $C_1 = [0, \infty] \times (0, \infty]$, then we have a form of hidden regular variation since

$$C_1 \subset C_2.$$

7.3. Form of the limit.

Assume μ is not a product and can standardize X

$$tP\left[\frac{\beta^{\leftarrow}(X)}{t} \le x, \frac{Y}{t} > y\right] \to \mu\big([0,\psi_2(x)] \times (y,\infty]\big) = \mu_*([0,x] \times (y,\infty])$$

on $C_1 = [0,\infty] \times (0,\infty]$. This is standard regular variation on the cone C_1 so

$$\mu_*(c\Lambda) = c^{-1}\mu_*(\Lambda).$$

$\exists$ spectral form: Let

$$\|(x,y)\| = x + y, \quad \aleph = \{(w, 1-w) : 0 \le w < 1\}$$

and

$$\mu_*\{\boldsymbol{x} : \|\boldsymbol{x}\| > r, \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \in A\} = r^{-1}S(A),$$

where S is a measure on $[0,1)$.
Conclude: Can write $\mu_*[0,x] \times (y,\infty]$ as function of S and get characterization of the class of limit measures.

7.4. Random norming.

When both variables can be standardized

$$tP\Big[\Big(\frac{\beta^{\leftarrow}(X)}{Y}, \frac{Y}{t}\Big) \in \cdot\Big] \to G \times \nu_1$$

in $M_+([0,\infty] \times (0,\infty])$ where

$$\nu_1(x,\infty] = x^{-1}, \quad G(x) = \int_{[0,\frac{x}{1+x}]} (1-w) S(dw).$$

8. Medical Care in Copenhagen

What to expect if you have a knee problem in Copenhagen:

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