Topics in Probability
Columbia University - Statistics
G8201D
Lecture 1

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Part I: Basics and Martingales

Basic Definitions and Preliminaries

- We assume as given a complete probability space $(\Omega, \mathcal{F}, P)$
- We are given a *filtration* $(\mathcal{F}_t)_{0 \leq t \leq \infty}$
- A **filtration** is a family of σ-algebras $(\mathcal{F}_t)_{0 \leq t \leq \infty}$ that is increasing: $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \leq t$.
- A **filtered complete probability space** $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{0 \leq t \leq \infty})$ is said to satisfy the **usual hypotheses** if
  1. (i)$\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$;
  2. (ii)$\mathcal{F}_t = \cap_{u>t} \mathcal{F}_u$, all t, $0 \leq t < \infty$; that is, the filtration $(\mathcal{F}_t)_{0 \leq t \leq \infty}$ is **right continuous**
- We always assume that the usual hypotheses hold

- A random variable $T : \Omega \to [0, \infty]$ is a **stopping time** if the event $\{T \leq t\} \in \mathcal{F}_t$, every t, $0 \leq t \leq \infty$.
- **Theorem:** The event $\{T < t\} \in \mathcal{F}_t$, $0 \leq t \leq \infty$, if and only if T is a stopping time.
- **Proof:** Since $\{T \leq t\} = \bigcap_{t+\epsilon > u > t} \{T < u\}$, any $\epsilon > 0$, we have $\{T \leq t\} \in \bigcap_{u > t} \mathcal{F}_u = \mathcal{F}_t$, so T is a stopping time
- For the converse, $\{T < t\} = \bigcup_{t > \epsilon > 0} \{T \leq t - \epsilon\}$, and $\{T \leq t - \epsilon\} \in \mathcal{F}_{t-\epsilon}$, hence also in $\mathcal{F}_t$. $\square$
- A **stochastic process** X on $(\Omega, \mathcal{F}, P)$ is a collection of random variables $(X_t)_{0 \leq t < \infty}$.
- The process X is said to be **adapted** if $X_t \in \mathcal{F}_t$ (that is, $\mathcal{F}_t$-measurable) for each t. We must take care to be precise about the concept of equality of two stochastic processes.

- Two stochastic processes X and Y are **modifications** if $X_t = Y_t$ a.s., each t.
- Two processes X and Y are **indistinguishable** if a.s., for all t, $X_t = Y_t$.
- If X and Y are *modifications* there exists a null set, N_t, such that if $\omega \notin N_t$, then $X_t(\omega) = Y_t(\omega)$. The null set N_t depends on t. Since the interval $[0, \infty)$ is uncountable the set $N = \underset{0 \leq t < \infty}{\rightarrow} \bigcup N_t$ could have any probability between 0 and 1, and it could even be non-measurable.
- If X and Y are *indistinguishable*, however, then there exists one null set N such that if $\omega \notin N$, then $X_t(\omega) = Y_t(\omega)$, for all t.

- A stochastic process X is said to be **càdlàg** if it a.s. has sample paths which are right continuous, with left limits.
- (The nonsensical word *càdlàg* is an acronym from the French "continu à droite, limites à gauche".)
- **Theorem:** Let X and Y be two stochastic processes, with X a modification of Y. If X and Y have right continuous paths a.s., then X and Y are indistinguishable.
- Càdlàg processes provide natural examples of stopping times
- Let X be a stochastic process and let Λ be a Borel set in $\mathbb{R}$. Define

$$T(\omega) = \inf\{t > 0 : X_t \in \Lambda\}.$$

Then T is called a **hitting time of Λ for X**

- **Theorem:** Let X be an adapted càdlàg stochastic process; and let Λ be an open set. Then the hitting time of Λ is a stopping time.
- **Proof:** It suffices to show that $\{T < t\} \in \mathcal{F}_t$, $0 \le t < \infty$. But

$$\{T < t\} = \bigcup_{s \in \mathbb{Q} \cap [0,t)} \{X_s \in \Lambda\},$$

since Λ is open and X has right continuous paths.
- Since $\{X_s \in \Lambda\} = X_s^{-1}(\Lambda) \in \mathcal{F}_s$, the result follows. $\square$
- For an adapted, càdlàg process the hitting times of open sets and of closed sets are stopping times

More on stopping times

- **Theorem 5 [Stability Properties of Stopping Times]:** Let S, T be stopping times. Then the following are stopping times:
  1. $S \wedge T = \min(S, T)$
  2. $S \vee T = \max(S, T)$
  3. $S + T$
  4. αS, *where* $\alpha > 1$
- Let T be a stopping time. **The stopping time σ-algebra, $\mathcal{F}_T$**, is defined to be

$$\{\Lambda \in \mathcal{F} : \Lambda \cap \{T \leq t\} \in \mathcal{F}_t, \text{ all } t \geq 0\}.$$

- **Theorem 6:** Let T be a finite stopping time. Then $\mathcal{F}_T$ is the smallest σ-algebra containing all càdlàg processes sampled at T. That is,

$$\mathcal{F}_T = \sigma\{X_T; X \text{ all adapted càdlàg processes}\}.$$

- **Theorem 7:** Let X be adapted and càdlàg. If $\Delta X_T 1_{\{T<\infty\}} = 0$ a.s. for each stopping time T, then ΔX is indistinguishable from the zero process.
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- **Corollary:** Let X and Y be adapted and càdlàg. If $\Delta X_T 1_{\{T<\infty\}}$ $= \Delta Y_T 1_{\{T<\infty\}}$ a.s. for each stopping time T, then ΔX and ΔY are indistinguishable.

Martingales

- A real valued, adapted process $X = (X_t)_{0 \leq t < \infty}$ is called a **martingale** (resp. **supermartingale, submartingale**) with respect to the filtration $(\mathcal{F}_t)_{0 \leq t \leq \infty}$ if
  1. $X_t \in L^1(dP)$; that is, $E\{|X_t|\} < \infty$;
  2. If $s \leq t$, then $E\{X_t | \mathcal{F}_s\} = X_s$, a.s. (resp. $E\{X_t | \mathcal{F}_s\} \leq X_s$, resp. $\geq X_s$)
- Martingales are only defined on $[0, \infty)$; that is, for finite t and not $t = \infty$. It is often possible to extend the definition to $t = \infty$
- A martingale X is said to be **closed** by a random variable Y if $E\{|Y|\} < \infty$ and $X_t = E\{Y | \mathcal{F}_t\}$, $0 \leq t < \infty$
- Note that an r.v. Y closing a martingale is not necessarily unique.

- **Theorem:** Let X be a supermartingale. The function $t \to E\{X_t\}$ is right continuous if and only if there exists a modification, Y, of X, which is càdlàg. Such a modification is unique, up to indistinguishability
- The above theorem is proved using Doob's Upcrossing Inequalities. If X is a martingale then $t \to E\{X_t\}$ is constant, and hence it has a right continuous modification.

The Martingale Convergence Theorem

- The Martingale Convergence Theorem is one of the major theorems of real analysis
- **Theorem [Martingale Convergence Theorem]:** Let X be a right continuous supermartingale, $\sup_{0 \le t < \infty} E\{|X_t|\} < \infty$. Then the random variable $Y = \lim_{t \to \infty} X_t$ a.s. exists, and $E\{|Y|\} < \infty$
- Moreover if X is a martingale closed by a random variable Z, then Y also closes X and $Y = E\{Z | \bigvee_{0 \le t < \infty} \mathcal{F}_t\}$
- A condition known as uniform integrability is sufficient for a martingale to be closed.

- A family of random variables $(U_\alpha)_{\alpha\in A}$ is **uniformly integrable** if

$$\lim_{n\to\infty} \sup \int_{\{|U_\alpha|\geq n\}} |U_\alpha| dP = 0.$$

- **Theorem:** Let $(U_\alpha)_{\alpha\in A}$ be a subset of L^1. The following are equivalent:
    1. $(U_\alpha)_{\alpha\in A}$ is uniformly integrable.
    2. $\sup_{\alpha\in A} E\{|U_\alpha|\} < \infty$, and whatever $\epsilon > 0$ there exists $\delta > 0$ such that $\Lambda \in \mathcal{F}$, $P(\Lambda) \leq \delta$, imply $E\{|U_\alpha 1_\Lambda|\} < \epsilon$.
    3. There exists a positive, increasing, convex function $G(x)$ defined on $[0,\infty)$ such that $\lim_{x\to\infty} \frac{G(x)}{x} = +\infty$ and $\sup\ E\{G \circ |U_\alpha|\} < \infty$. The assumption that G is convex is not needed for the implications (iii)$\Rightarrow$(ii) and (iii)$\Rightarrow$(i).

Doob's Optional Sampling Theorem

- **Theorem:** Let X be a right continuous martingale which is uniformly integrable. Then $Y = \lim_{t\to\infty} X_t$ a.s. exists, $E\{|Y|\} < \infty$, and Y closes X as a martingale.
- **Theorem:** Let X be a (right continuous) martingale. Then $(X_t)_{t\geq 0}$ is uniformly integrable if and only if $Y = \lim_{t\to\infty} X_t$ exists a.s., $E\{|Y|\} < \infty$, and $(X_t)_{0\leq t\leq\infty}$ is a martingale, where $X_\infty = Y$
- The next theorem is one of the most useful martingale theorems for our purposes.
- **Theorem [Doob's Optional Sampling Theorem]:** Let X be a right continuous martingale, which is closed by a random variable X_∞. Let S and T be two stopping times such that $S \leq T$ a.s. Then X_S and X_T are integrable and

$$X_S = E\{X_T|\mathcal{F}_S\} \qquad \text{a.s.}$$

- Doob's Optional Sampling Theorem has a similar version for supermartingales.
- **Theorem:** Let X be a right continuous supermartingale (martingale), and let S and T be two bounded stopping times such that $S \leq T$ a.s. Then X_S and X_T are integrable and

$$X_S \geq E\{X_T | \mathcal{F}_S\} \qquad \text{a.s. } (=).$$

- If T is a stopping time, then so is $t \wedge T = \min(t, T)$, for each $t \geq 0$
- Let X be a stochastic process and let T be a random time. X^T is said to be **the process stopped at T** if $X_t^T = X_{t \wedge T}$.

- If X is adapted and càdlàg and if T is a stopping time, then

$$X_t^T = X_{t \wedge T} = X_t 1_{\{t < T\}} + X_T 1_{\{t \geq T\}}$$

  is also adapted.
- A martingale stopped at a stopping time is still a martingale, as the next theorem shows.
- **Theorem:** Let X be a uniformly integrable right continuous martingale, and let T be a stopping time. Then $X^T = (X_{t \wedge T})_{0 \leq t \leq \infty}$ is also a uniformly integrable right continuous martingale.

Jensen's Inequality

- **Corollary:** Let Y be an integrable random variable and let S, T be stopping times. Then

$$E\{E\{Y|\mathcal{F}_S\}|\mathcal{F}_T\} = E\{E\{Y|\mathcal{F}_T\}|\mathcal{F}_S\}$$
$$= E\{Y|\mathcal{F}_{S\wedge T}\}.$$

- **Theorem [Jensen's Inequality]:** Let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex, and let X and $\varphi(X)$ be integrable random variables. For any σ-algebra $\mathcal{G}$,

$$\varphi \circ E\{X|\mathcal{G}\} \leq E\{\varphi(X)|\mathcal{G}\}.$$

- **Corollary:** Let X be a martingale, and let φ be convex such that $\varphi(X_t)$ is integrable, $0 \leq t < \infty$. Then $\varphi(X)$ is a submartingale. In particular, if M is a martingale, then $|M|$ is a submartingale.
- **Corollary:** Let X be a submartingale and let φ be convex, nondecreasing, and such that $\varphi(X_t)_{0\leq t<\infty}$ is integrable. Then $\varphi(X)$ is also a submartingale.

Doob's Martingale Inequalities

- **Theorem:** Let X be a positive submartingale. For all $p > 1$, with q conjugate to p (i.e., $\frac{1}{p} + \frac{1}{q} = 1$), we have

$$\| \sup_t |X_t| \|_{L^p} \leq q \sup_t \|X_t\|_{L^p} .$$

- **We let X^* denote** $\sup_s |X_s|$. Note that if M is a martingale with $M_\infty \in L^2$, then $|M|$ is a positive submartingale, and taking $p = 2$ we have

$$E\{(M^*)^2\} \leq 4E\{M_\infty^2\}.$$

Local Martingales

- For a process X and a stopping time T, X^T denotes the *stopped process*

$$X_t^T = X_{t \wedge T} = X_t 1_{\{t < T\}} + X_T 1_{\{t \geq T\}}$$

- An adapted, càdlàg process X is a **local martingale** if there exists a sequence of increasing stopping times, T_n, with $\lim_{n \to \infty} T_n = \infty$ a.s. such that $X_{t \wedge T_n} 1_{\{T_n > 0\}}$ is a uniformly integrable martingale for each n. Such a sequence (T_n) of stopping times is called a **fundamental sequence**
- **Example 1:** Any càdlàg martingale is a local martingale (take $T_n \equiv n$)

- **Example 2:** Let $(B_t)_{0 \leq t < \infty}$ be a Brownian motion in $\mathbb{R}^3$ with $B_0 = x$, where $x \neq 0$. Let $u(y) = \frac{1}{\|y\|}$, a superharmonic function on $\mathbb{R}^3$. As a consequence of Itô's Formula (to be shown later) one can show that $X_t = u(B_t)$ is a positive supermartingale (indeed, it is even a uniformly integrable supermartingale)
- Next let $T_n = \inf\{t > 0 : \|B_t\| \leq \frac{1}{n}\}$. Outside of the ball of radius $\frac{1}{n}$ centered at the origin the function u is harmonic
- Again by Itô's Formula one can show that this implies $u(B_{t \wedge T_n})$ is a martingale. Since $u(B_{t \wedge T_n})$ is bounded by n it is uniformly integrable. On the other hand, if $B_0 = x \neq 0$ then $\lim_{t \to \infty} E\{u(B_t)\} = 0$ as is easily seen by a calculation, while $E\{u(B_0)\} = \frac{1}{\|x\|}$. Since the expectations of a martingale are constant, $u(B_t)$ is not a martingale.

- The reason we multiply $X_{t\wedge T_n}$ by $1_{\{T_n>0\}}$ is to relax the integrability condition on X_0.
- A stopping time T **reduces** a process M if M^T is a uniformly integrable martingale
- **Theorem:** Let M, N be local martingales and let S and T be stopping times
  1. If T reduces M and $S \leq T$ a.s., then S reduces M;
  2. The sum $M+N$ is also a local martingale;
  3. If S, T both reduce M, then $S \vee T$ also reduces M;
  4. The processes M^T, $M^T 1_{\{T>0\}}$ are local martingales;
  5. Let X be a càdlàg process and let T_n be a sequence of stopping times increasing to ∞ a.s. such that $X^{T_n} 1_{\{T_n>0\}}$ is a local martingale for each n. Then X is a local martingale

- **Corollary:** Local martingales form a vector space.
- **Theorem:** Let X be a local martingale such that $E\{X_t^*\} < \infty$ for every $t \geq 0$. Then X is a martingale. If $E\{X^*\} < \infty$, then X is a uniformly integrable martingale

The "Classical" Change of Variables Formula of Stieltjes

- A process is called an FV process if it has sample paths which are a.s. of finite variation on compact time sets
- **Theorem [Change of Variables]:** Let A be an FV process with continuous paths, and let f be such that its derivative f' exists and is continuous. Then $(f(A_t))_{t\geq 0}$ is an FV process and

$$f(A_t) - f(A_0) = \int_0^t f'(A_s)dA_s$$

- **Proof:** For fixed ω, the function $s \to f'(A_s(\omega))$ is continuous on $[0, t]$ and hence bounded. Therefore the integral $\int_0^t f'(A_s) dA_s$ exists.
- Fix t and let π_n be a sequence of partitions of $[0, t]$ with $\lim_{n\to\infty} \text{mesh}(\pi_n) = 0$. Then

$$
\begin{aligned}
f(A_t) - f(A_0) &= \sum_{t_k, t_{k+1} \in \pi_n} \{f(A_{t_{k+1}}) - f(A_{t_k})\} \\
&= \sum_k f'(A_{S_k})(A_{t_{k+1}} - A_{t_k}),
\end{aligned}
$$

  by the Mean Value Theorem, for some S_k, $t_k \leq S_k \leq t_{k+1}$
- The result now follows by taking limits. $\square$

Change of Variables and u substitution in Calculus

- Integrate $\int_{-2}^{0} x\sqrt{2x^2+1}dx$
- Make the u substitution $u = 2x^2 + 1$; $du = -4xdx$
- Recall the formula from Calculus

$$\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$$

- Then

$$\int_{-2}^{0} x\sqrt{2x^2+1}dx = \int_9^1 \sqrt{u}\frac{1}{4}du = -\frac{13}{3}$$

- We have made the change of variable where $F(y) = \int_0^y f(u)du$
- and

$$\begin{aligned} F(u(t)) &= \int_0^t f(u(s))du(s) \\ &= \int_0^t f(u(s))u'(s)ds \end{aligned}$$

Part II: Stochastic Integration

Naïve Stochastic Integration is Impossible

- First we "recall" the Banach-Steinhaus theorem
- It is also known as the Principle of Uniform Boundedness
- **Theorem:** Let X be a Banach space and let Y be a normed linear space. Let $\{T_\alpha\}$ be a family of bounded linear operators from X into Y. If for each $x \in X$ the set $\{T_\alpha x\}$ is bounded, then the set $\{\|T_\alpha\|\}$ is bounded

- Let $x(t)$ be a right continuous function on $[0,1]$, and let π_n be a refining sequence of dyadic rational partitions of $[0,1]$ with $\lim_{n\to\infty}\ \mathrm{mesh}(\pi_n) = 0$
- What conditions on x are needed so that the sums

$$S_n = \sum_{t_k, t_{k+1} \in \pi_n} h(t_k)(x(t_{k+1}) - x(t_k))$$

converge to a finite limit as $n \to \infty$ for all continuous functions h?

- **Theorem:** If the sums S_n converge to a limit for every continuous function h then x is of finite variation
- Suppose then we want to construct a stochastic integral using Riemann sums (N. Wiener)
- For example, suppose $s \mapsto X(s, \omega)$ is right continuous (or even continuous), a.s., can we have the sums

$$\sum_{t_k, t_{k+1} \in \pi_n[0,1]} H_{t_k}(X_{t_{k+1}} - X_{t_k})$$

  converging to a limit *in probability* for every continuous process H?
- Unfortunately the answer is that X must still have paths of finite variation, a.s.
- Brownian motion, and continuous martingales in general, have paths of infinite variation on compact time sets

Semimartingales

- A process H is said to be **simple predictable** if H has a representation

$$H_t = H_0 1_{\{0\}}(t) + \sum_{i=1}^{n} H_i 1_{(T_i, T_{i+1}]}(t)$$

  where $0 = T_1 \leq \cdots \leq T_{n+1} < \infty$ is a finite sequence of stopping times, $H_i \in \mathcal{F}_{T_i}$ with $|H_i| < \infty$ a.s., $0 \leq i \leq n$
- The collection of simple predictable processes is denoted **S**

- Let X be a stochastic process. An operator, I_X induced by X should have two fundamental properties to earn the name "integral"
    1. The operator I_X should be linear
    2. The operator should satisfy some version of the Bounded Convergence Theorem
- A particularly weak form of the Bounded Convergence Theorem is that the uniform convergence of processes H^n to H implies only the convergence in probability of $I_X(H^n)$ to $I_X(H)$