Using Decomposition Methods to Solve Pricing Problems in Network Revenue Management

(Research Paper)

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Abstract

In this paper, we develop two methods for making pricing decisions in network revenue management problems. We consider a setting where the probability of observing a request for an itinerary depends on the prices and the objective is to dynamically adjust the prices so as to maximize the total expected revenue. The idea behind both of our methods is to decompose the dynamic programming formulation of the pricing problem by the flight legs and to obtain value function approximations by focusing on one flight leg at a time. We show that our methods provide upper bounds on the optimal total expected revenue and these upper bounds are tighter than the one provided by a deterministic linear program commonly used in practice. Our computational experiments yield two important results. First, our methods provide substantial improvements over the deterministic linear program. The average gap between the total expected revenues obtained by our methods and the deterministic linear program is 7.11%. On average, our methods tighten the upper bounds obtained by the deterministic linear program by 3.66%. Second, the two methods that we develop have different strengths. In particular, while one method is able to obtain tighter upper bounds, the other one is able to obtain pricing policies that yield higher total expected revenues.

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Capacity allocation has traditionally been regarded as the prevalent control policy for the network revenue management systems operated by the airlines. In particular, a capacity allocation policy fixes the prices for the itineraries at prespecified levels and decides which itineraries to close and which itineraries to keep open for sale so as to maximize the total expected revenue. It has been argued that the airlines are suitable for capacity allocation since their promotion and administrative needs require them to fix the prices for the itineraries in advance of the sales and a capacity allocation policy indeed allows them to work with fixed prices. However, this argument has started to lose its validity with the advent of online sales channels allowing the airlines to dynamically adjust the prices for the itineraries as the sales take place. As a result, pricing has started to emerge as a feasible control policy for the network revenue management systems operated by the airlines.

One of the traditional approaches for making pricing decisions in network revenue management problems is based on a deterministic linear program. This deterministic linear program assumes that the arrivals of the itinerary requests are given by deterministic functions of the prices. The decision variables correspond to the numbers of time periods in the planning horizon for which we charge the different price levels for the itineraries. The deterministic linear program dates back to the work of Gallego and van Ryzin (1997) and it has received a lot of attention from academics and practitioners over the years, but due to its deterministic nature, it is not able to capture the temporal dynamics of the itinerary requests accurately.

In this paper, we propose two new methods suitable for making pricing decisions in network revenue management problems. In the setting we consider, the probability of observing a request for an itinerary depends on the prices that we charge for the itineraries and the objective is to dynamically adjust the prices so as to maximize the total expected revenue. Both of the methods that we propose use the deterministic linear program mentioned above as a starting point. In particular, by using the dual solution to the deterministic linear program, we first allocate the immediate revenue associated with a certain price level among the different flight legs. Once we have allocated the immediate revenue associated with a certain price level among the different flight legs, we can solve a sequence of revenue management problems, each taking place over a single flight leg. In the single leg revenue management problem that takes place over a particular flight leg, if we charge a certain price level for a certain itinerary, then the revenue that we obtain is given by the portion of the price level that is allocated to the flight leg. By solving the dynamic programming formulation of the single leg revenue management problem for each flight leg, we obtain a value function from each one of the flight legs, in which case, we sum up these value functions to obtain a value function approximation for the original network revenue management problem. Ultimately, both of the methods that we propose construct separable approximations to the value functions.

The methods that we propose in this paper provide advantages when compared with the deterministic linear program. To begin with, since our methods use dynamic programming formulations for the single leg revenue management problems, they are likely to capture the temporal dynamics of the itinerary requests more accurately than the deterministic linear program. In addition, it is possible to show that the deterministic linear program provides an upper bound on the total expected revenue obtained by the optimal control policy. Such upper bounds become useful when we assess the optimality gap of a suboptimal control policy. We show that our methods also obtain upper bounds on the optimal total expected revenue and the upper bounds obtained by our methods are provably tighter than those from the deterministic linear program. Finally, our computational experiments demonstrate that the two methods that we propose can provide substantial improvements over the deterministic linear program. Averaging over all of the test problems in our experimental setup, the gap between the total expected revenues obtained by our methods and the deterministic linear program is 7.11%, whereas the gap between the upper bounds obtained by our methods and the deterministic linear program is 3.66%.

The basic idea behind the two methods that we propose is to decompose the pricing problem over an airline network by the flight legs and to obtain value function approximations by solving a sequence of single leg revenue management problems. Therefore, our methods can be visualized as dynamic programming decomposition approaches. There are some other dynamic programming decomposition approaches in the literature that try to construct good control policies by focusing on one flight leg at a time. However, these approaches exclusively use capacity control policies, whereas our focus is on pricing. To our knowledge, there are few practical algorithms for pricing and the transition from capacity control to pricing is nontrivial and practically important. Zhang and Adelman (2009) were the first to show that a dynamic programming decomposition approach can provide upper bounds on the optimal total expected revenue in the capacity allocation setting. Liu and van Ryzin (2008) use dynamic programming decomposition approaches to model the customer choice behavior, where each customer observes the set of itineraries that are available for sale and makes a choice among them. Topaloglu (2009) demonstrates that it is possible to develop dynamic programming decomposition approaches by using a suitable Lagrangian relaxation argument on the dynamic programming formulation of a capacity allocation problem. Erdelyi and Topaloglu (2009) follow a dynamic programming decomposition idea to develop a joint capacity allocation and overbooking model.

Although pricing is a fundamental control mechanism in network revenue management, most of the pricing papers in the literature focus on pricing a single product in isolation, whereas the network revenue management setting requires pricing multiple itineraries that interact with each other. Gallego and van Ryzin (1994) analyze the problem of dynamically adjusting the price of a single product and characterize the form of the optimal policy. They also show that a single price policy is asymptotically optimal as the initial inventory of the product and the length of the selling horizon increase linearly with the same rate. Feng and Gallego (2000), Feng and Xiao (2000) and Zhao and Zheng (2000) extend the analysis in Gallego and van Ryzin (1994) to incorporate more complicated demand dynamics and pricing constraints. Feng and Gallego (1995) consider the case where the price of a product can be adjusted only once, either from high to low or from low to high. They characterize the optimal timing of the price change. Maglaras and Meissner (2006) establish that certain pricing problems can be converted into equivalent capacity allocation problems and this result immediately allows them to extend the structural properties for capacity allocation problems to pricing problems.

The literature is thinner when we focus on pricing over an airline network. Gallego and van Ryzin (1997) propose a deterministic optimization problem for pricing multiple itineraries that interact with each other. They show that the pricing decisions made by this deterministic optimization problem are asymptotically optimal in the same sense as in Gallego and van Ryzin (1994). We use a variant of their approach as a benchmark strategy in our computational experiments. Kleywegt (2001) develops a joint pricing and overbooking model, where the itinerary requests are deterministic functions of the prices and he solves the model by using Lagrangian duality arguments. Zhang and Cooper (2009) consider the problem of pricing substitutable flights that operate between the same origin destination pair. They build upper and lower bounds on the value functions and use these bounds to construct pricing policies, but their approach does not appear to extend to a general airline network. Kunnumkal and Topaloglu (2009) propose a stochastic approximation algorithm for making pricing decisions over an airline network. The review papers by McGill and van Ryzin (1999), Bitran and Caldentey (2003) and Elmaghraby and Keskinocak (2003) and the book by Talluri and van Ryzin (2005) provide extensive coverage of pricing models in network revenue management.

In this paper, we make the following research contributions. 1) We develop two methods for making pricing decisions in network revenue management problems. Our methods are based on decomposing the pricing problem over an airline network by the flight legs and obtaining value function approximations by focusing on one flight leg at a time. Since our methods use dynamic programming formulations, they capture the temporal dynamics of the itinerary requests more accurately than the deterministic linear program mentioned above. 2) We show that both of our methods provide upper bounds on the total expected revenue obtained by the optimal control policy. It is possible to show that the deterministic linear program also provides such an upper bound, but the upper bounds provided by our methods are provably tighter than those provided by the deterministic linear program. 3) Our computational experiments demonstrate that the methods that we propose can provide substantial improvements over the deterministic linear program. On average, the total expected revenues obtained by our methods improve those obtained by the deterministic linear program by 7.11% and there are test problems where the performance gap can be as high as 17.02%. Similarly, the average gap between the upper bounds obtained by our methods and the deterministic linear program is 3.66% and there are test problems where the gap between the upper bounds is as high as 7.98%. Furthermore, our computational experiments indicate that the two methods that we propose complement each other as they provide improvements over the deterministic linear program. In particular, one of the methods is successful in obtaining tight upper bounds on the optimal total expected revenue, whereas the other method is successful in identifying pricing policies that yield high total expected revenues.

The rest of the paper is organized as follows. In Section 1, we formulate the pricing problem over an airline network as a dynamic program. In Section 2, we describe the deterministic linear program, show that it provides an upper bound on the optimal total expected revenue and demonstrate how it can be used to construct a pricing policy. In Sections 3 and 4, we propose two new methods for making pricing decisions. In each one of these sections, we focus on one of the two methods and show that the method in question provides an upper bound on the optimal total expected revenue and this upper bound is tighter than the one provided by the deterministic linear program. Furthermore, we demonstrate how our methods can be used to construct pricing policies. In Section 5, we provide computational experiments. In Section 6, we conclude.

1 Problem Formulation

We have a set of flight legs that can be used to serve the requests for itineraries that arrive randomly over time. At each time period, we adjust the prices for the itineraries, which, in turn, determine the probability of observing a request for an itinerary. Pricing serves as the only control mechanism and we do not have the option of rejecting an itinerary request. Whenever there is an itinerary request, we accept the itinerary request, generate a revenue that reflects the price for the itinerary and consume the capacities on the relevant flight legs. The objective is to adjust the prices for the itineraries over time so as to maximize the total expected revenue.

The problem takes place over the planning horizon $\mathcal{T} = \{1, \ldots, \tau\}$ and time period $\tau + 1$ is the departure time of the flight legs. A time period corresponds to a small enough interval of time that there is at most one itinerary request at each time period. The set of flight legs in the airline network is $\mathcal{L}$ and the set of itineraries is $\mathcal{J}$. The total available capacity on flight leg i is c_i. If we serve a request for itinerary j, then we consume a_{ij} units of capacity on flight leg i. The set of possible prices for itinerary j is given by the finite set $\{p_j^k : k \in \mathcal{K}\}$ and the price that we charge for itinerary j has to take a value in this set. Having a finite set of possible prices is not very restrictive from practical perspective as the cardinality of the set $\mathcal{K}$ can be quite large. If we charge the price p_j^k for itinerary j, then we observe a request for itinerary j at a time period with probability λ_j^k. For notational brevity, we let $r_j^k = \lambda_j^k \, p_j^k$ so that r_j^k is the expected revenue that we generate at a time period from itinerary j when we charge the price p_j^k for this itinerary. We employ a few assumptions for the price and probability pairs $\{(p_j^k, \lambda_j^k) : k \in \mathcal{K}\}$. First, we assume that $\sum_{j \in \mathcal{J}} \max_{k \in \mathcal{K}} \{\lambda_j^k\} \leq 1$, so that irrespective of the prices that we charge, there is at most one itinerary request at each time period. Second, we assume that there exists some $\phi \in \mathcal{K}$ such that $\lambda_j^\phi = 0$. In this case, if we do not have enough capacity to serve a request for itinerary j, then we can charge the price p_j^ϕ to ensure that we do not observe a request for itinerary j. Third, as evident from our notation, we assume that the probability of observing a request for itinerary j depends only on the price for itinerary j, but not on the prices for the other itineraries. This assumption is reasonable when the itineraries do not serve as substitutes of each other. Furthermore, we point out possible relaxations of this assumption throughout the paper.

We use x_{it} to denote the remaining capacity on flight leg i at the beginning of time period t so that $x_t = \{x_{it} : i \in \mathcal{L}\}$ gives the state of the remaining leg capacities. We use $u_t = \{u_{jt}^k : j \in \mathcal{J},\ k \in \mathcal{K}\}$ to capture the decisions at time period t, where $u_{jt}^k = 1$ if we charge the price p_j^k for itinerary j at time period t and $u_{jt}^k = 0$ otherwise. In this case, the set of feasible decisions at time period t is given by

$$\mathcal{U}(x_t) = \Big\{ u_t \in \{0,1\}^{|\mathcal{J}||\mathcal{K}|} : \sum_{k \in \mathcal{K}} a_{ij} \, \lambda_j^k \, u_{jt}^k \leq x_{it} \qquad \forall \, i \in \mathcal{L},\ j \in \mathcal{J} \tag{1}$$

$$\sum_{k \in \mathcal{K}} u_{jt}^k = 1 \qquad \forall j \in \mathcal{J} \Big\}. \tag{2}$$

Noting that $\lambda_j^\phi = 0$, constraints (1) ensure that if we do not have enough capacity to serve a request for itinerary j, then we charge the price p_j^ϕ for this itinerary. Constraints (2) ensure that each itinerary is offered at a single price at each time period. We use $V_t(x_t)$ to denote the maximum total expected revenue that can be obtained over the time periods $\{t, \ldots, \tau\}$ given that the state of the remaining leg capacities at the beginning of time period t is x_t. Letting e_i be the $|\mathcal{L}|$ dimensional unit vector with a one in the element corresponding to flight leg i, we can evaluate the value functions $\{V_t(\cdot) : t \in \mathcal{T}\}$ by solving the optimality equation

$$V_t(x_t) = \max_{u_t \in \mathcal{U}(x_t)} \left\{ \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} u_{jt}^k \left\{ r_j^k + \lambda_j^k \, V_{t+1}(x_t - \textstyle\sum_{i \in \mathcal{L}} e_i \, a_{ij}) \right\} + \left[1 - \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} u_{jt}^k \, \lambda_j^k \right] V_{t+1}(x_t) \right\} \quad (3)$$

with the boundary condition that $V_{\tau+1}(\cdot) = 0$. If the state of the remaining leg capacities at the beginning of time period t is given by x_t, then we can find the optimal pricing decisions by solving the problem on the right side of the optimality equation in (3).

Unfortunately, the optimality equation in (3) involves a high dimensional state variable and the computation of $\{V_t(\cdot) : t \in \mathcal{T}\}$ easily gets intractable for practical problems. In the next section, we begin by formulating a linear programming approximation to the optimality equation in (3). This linear program later serves as a starting point for our solution methods.

2 Deterministic Linear Program

Under the assumption that the itinerary requests take on their expected values, it is possible to formulate a deterministic linear program to approximate the total expected revenue over the planning horizon. In particular, letting w_j^k be the number of time periods at which we charge the price p_j^k for itinerary j, we can solve the problem

$$\max \quad \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} r_j^k \, w_j^k \quad (4)$$

$$\text{subject to} \quad \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} a_{ij} \, \lambda_j^k \, w_j^k \le c_i \qquad \forall \, i \in \mathcal{L} \quad (5)$$

$$\sum_{k \in \mathcal{K}} w_j^k = \tau \qquad \forall \, j \in \mathcal{J} \quad (6)$$

$$w_j^k \ge 0 \qquad \forall \, j \in \mathcal{J}, \; k \in \mathcal{K} \quad (7)$$

to approximate the total expected revenue. In the problem above, the objective function accounts for the total expected revenue over the planning horizon. Constraints (5) ensure that our decisions do not violate the leg capacities. Constraints (6) ensure that the total number of time periods at which we charge the different prices is equal to the number of time periods in the planning horizon.

There are two uses of problem (4)-(7). First, the optimal objective value of problem (4)-(7) provides an upper bound on the total expected revenue obtained by the optimal control policy. Such an upper bound becomes useful when assessing the optimality gap of a suboptimal control policy. In particular, using $c = \{c_i : i \in \mathcal{L}\}$ to denote the vector of available capacities on the flight legs, $V_1(c)$ is the optimal total expected revenue over the planning horizon. In this case, letting $\hat{z}_{LP}$ be the optimal objective value of problem (4)-(7), the next proposition shows that $\hat{z}_{LP}$ provides an upper bound on $V_1(c)$. The proofs of all of our results can be found in the appendix.

Proposition 1 *We have $V_1(c) \le \hat{z}_{LP}$.*

Gallego and van Ryzin (1997) show an analogue of Proposition 1. The appealing aspect of their result is that they assume that the prices can take values over a continuum and the itinerary requests arrive in continuous time according a Poisson process. However, their result requires that the expected revenue from an itinerary is a concave function of the arrival probability, whereas Proposition 1 does not make an assumption for the relationship between r_j^k and λ_j^k.

A second use of problem (4)-(7) occurs when we try to make the pricing decisions. In particular, if we let $\{\hat{w}_j^k : j \in \mathcal{J},\ k \in \mathcal{K}\}$ be the optimal solution to problem (4)-(7), then one alternative for making the pricing decisions is to charge the price p_j^k for itinerary j with probability $\hat{w}_j^k / \tau$ at each time period. If we do not have enough capacity to serve a request for itinerary j, then we naturally charge the price p_j^ϕ for itinerary j. We refer to this decision rule as DLP-P, where DLP stands for deterministic linear program and P stands for primal. Another alternative for making the pricing decisions is to use the optimal dual solution to problem (4)-(7). In particular, if we let $\{\hat{\pi}_i : i \in \mathcal{L}\}$ be the optimal values of the dual variables associated with constraints (5) in problem (4)-(7), then we can use $\hat{\pi}_i$ to capture the opportunity cost of a seat on flight leg i. This allows us to approximate the value functions $\{V_t(\cdot) : t \in \mathcal{T}\}$ with linear functions $\{\tilde{V}_t(\cdot) : t \in \mathcal{T}\}$ of the form $\tilde{V}_t(x_t) = \sum_{i \in \mathcal{L}} \hat{\pi}_i \, x_{it}$. In this case, we can replace the value functions $\{V_t(\cdot) : t \in \mathcal{T}\}$ on the right side of problem (3) with the linear value function approximations $\{\tilde{V}_t(\cdot) : t \in \mathcal{T}\}$ and solve this problem to make the pricing decisions at time period t. We refer to this decision rule as DLP-D, where D stands for dual.

Closing this section, we briefly elaborate on how to extend problem (4)-(7) to cover the case where the probability of observing a request for itinerary j does not depend only on the price for itinerary j, but also on the prices for the other itineraries. To cover this case, we let $\{p^k : k \in \mathcal{K}\}$ be the set of possible joint prices for the itineraries, where the vector $p^k = \{p_j^k : j \in \mathcal{J}\}$ includes the prices of all itineraries. If we charge the prices p^k for the itineraries, then we observe a request for itinerary j at a time period with probability λ_j^k and λ_j^k can depend on the whole vector of prices p^k. In this case, if we let w^k be the number of time periods at which we charge the prices p^k for the itineraries, then all we need to do is to replace the decision variables $\{w_j^k : j \in \mathcal{J}\}$ in problem (4)-(7) with a single decision variable w^k. In this case, the objective function of problem (4)-(7) becomes $\sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{J}} r_j^k \, w^k$, where the term $\sum_{j \in \mathcal{J}} r_j^k$ can be interpreted as the expected revenue that we generate at a time period from all itineraries when we charge the prices p^k. The first set of constraints become $\sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{J}} a_{ij} \, \lambda_j^k \, w^k \leq c_i$ for all $i \in \mathcal{L}$, where the term $\sum_{j \in \mathcal{J}} a_{ij} \, \lambda_j^k$ can be interpreted as the expected capacity consumption at a time period on flight leg i when we charge the prices p^k. The second set of constraints become $\sum_{k \in \mathcal{K}} w^k = \tau$. We note that the number of possible joint prices $\{p^k : k \in \mathcal{K}\}$ can be very large in a practical application, which implies that the number of decision variables in problem (4)-(7) can also be very large. However, the number of constraints in problem (4)-(7) is always manageable. Therefore, we can solve problem (4)-(7) in a tractable fashion by using column generation.

3 Decomposition by Revenue Allocation

A shortcoming of the DLP-P and DLP-D decision rules is that they are based on the assumption that the itinerary requests take on their expected values. In this section, we build on problem (4)-(7) to develop a decision rule that addresses the stochastic nature of the itinerary requests more accurately. We begin by augmenting the set of flight legs with a fictitious flight leg ψ so that the set of flight legs becomes $\mathcal{L} \cup \{\psi\}$. We assume that none of the itineraries use the fictitious flight leg so that its capacity is irrelevant. In this case, using the decision variables $\{w_{ij}^k : i \in \mathcal{L} \cup \{\psi\},\ j \in \mathcal{J},\ k \in \mathcal{K}\}$ instead of the decision variables $\{w_j^k : j \in \mathcal{J},\ k \in \mathcal{K}\}$, problem (4)-(7) is equivalent to the problem

$$\max \quad \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} r_j^k \, w_{\psi j}^k \tag{8}$$

$$\text{subject to} \quad \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} a_{ij} \, \lambda_j^k \, w_{ij}^k \leq c_i \qquad \forall \, i \in \mathcal{L} \tag{9}$$

$$\sum_{k \in \mathcal{K}} w_{ij}^k = \tau \qquad \forall \, i \in \mathcal{L},\ j \in \mathcal{J} \tag{10}$$

$$w_{\psi j}^k - w_{ij}^k = 0 \qquad \forall \, i \in \mathcal{L},\ j \in \mathcal{J},\ k \in \mathcal{K} \tag{11}$$

$$w_{ij}^k \geq 0 \qquad \forall \, i \in \mathcal{L},\ j \in \mathcal{J},\ k \in \mathcal{K}. \tag{12}$$

To see the equivalence between problems (4)-(7) and (8)-(12), we note that we can use constraints (11) to replace all of the decision variables $\{w_{ij}^k : i \in \mathcal{L}\}$ in problem (8)-(12) with a single decision variable $w_{\psi j}^k$. In this case, we can drop constraints (11) from problem (8)-(12) and problems (4)-(7) and (8)-(12) become equivalent to each other. Therefore, recalling the notation in Section 2, the optimal objective value of problem (8)-(12) is still $\hat{z}_{LP}$.

We let $\{\hat{\mu}_{ij}^k : i \in \mathcal{L},\ j \in \mathcal{J},\ k \in \mathcal{K}\}$ be the optimal values of the dual variables associated with constraints (11) in problem (8)-(12). If we dualize these constraints by associating the multipliers $\{\hat{\mu}_{ij}^k : i \in \mathcal{L},\ j \in \mathcal{J},\ k \in \mathcal{K}\}$ with them, then the objective function of problem (8)-(12) reads $\sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} [r_j^k - \sum_{i \in \mathcal{L}} \hat{\mu}_{ij}^k] \, w_{\psi j}^k + \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} \hat{\mu}_{ij}^k \, w_{ij}^k$. By the constraints in the dual of problem (8)-(12) associated with the decision variables $\{w_{\psi j}^k : j \in \mathcal{J},\ k \in \mathcal{K}\}$, we have $\sum_{i \in \mathcal{L}} \hat{\mu}_{ij}^k = r_j^k$ for all $j \in \mathcal{J}$, $k \in \mathcal{K}$. Therefore, the term $[r_j^k - \sum_{i \in \mathcal{L}} \hat{\mu}_{ij}^k]$ in the last expression is equal to zero and the optimal objective value of problem (8)-(12) is equal to the optimal objective value of the problem

$$\max \quad \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} \hat{\mu}_{ij}^k \, w_{ij}^k \tag{13}$$

$$\text{subject to} \quad (9), (10), (12). \tag{14}$$

The crucial observation here is that the objective function and all of constraints (9), (10) and (12) in problem (13)-(14) decompose by the flight legs. This implies that problem (13)-(14) decomposes into $|\mathcal{L}|$ subproblems and the subproblem corresponding to flight leg i has the form

$$\max \quad \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} \hat{\mu}_{ij}^k \, w_{ij}^k \tag{15}$$

$$\text{subject to} \quad \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} a_{ij} \, \lambda_j^k \, w_{ij}^k \leq c_i \tag{16}$$

$$\sum_{k \in \mathcal{K}} w_{ij}^k = \tau \qquad \forall \, j \in \mathcal{J} \tag{17}$$

$$w_{ij}^k \geq 0 \qquad \forall \, j \in \mathcal{J},\ k \in \mathcal{K}. \tag{18}$$

Summing up the discussion in the last two paragraphs, if we let $\hat{z}^i_{LP}$ be the optimal objective value of problem (15)-(18), then we have $\hat{z}_{LP} = \sum_{i \in \mathcal{L}} \hat{z}^i_{LP}$.

Comparing problem (15)-(18) with problem (4)-(7), we observe that problem (15)-(18) corresponds to the deterministic linear program for a revenue management problem that takes over the single flight leg i. In this single leg revenue management problem, if we charge the price p^k_j for itinerary j, then the expected revenue that we generate at a time period from itinerary j is given by $\hat{\mu}^k_{ij}$. Therefore, we can visualize $\hat{\mu}^k_{ij}$ as the portion of the expected revenue r^k_j that is allocated to flight leg i. Since by Proposition 1, the optimal objective value of the deterministic linear program provides an upper bound on the optimal total expected revenue, $\hat{z}^i_{LP}$ provides an upper bound on the optimal total expected revenue in the single leg revenue management problem that takes place over flight leg i.

On the other hand, we can compute the optimal total expected revenue in the single leg revenue management problem that takes place over flight leg i by solving the optimality equation

$$v^i_t(x_{it}) = \max_{u_t \in \mathcal{U}^i(x_{it})} \left\{ \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} u^k_{jt} \left\{ \hat{\mu}^k_{ij} + \lambda^k_j \, v^i_{t+1}(x_{it} - a_{ij}) \right\} + \left[1 - \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} u^k_{jt} \, \lambda^k_j \right] v^i_{t+1}(x_{it}) \right\} \tag{19}$$

with the boundary condition that $v^i_{\tau+1}(\cdot) = 0$. The optimality equation above is similar to the one in (3), but the state variable only keeps track of the remaining capacity on flight leg i. The superscript i in the value functions emphasizes that the optimality equation above computes the optimal total expected revenue for the single leg revenue management problem that takes place over flight leg i. The set of feasible decisions $\mathcal{U}^i(x_{it})$ is given by

$$\mathcal{U}^i(x_{it}) = \Big\{ u_t \in \{0,1\}^{|\mathcal{J}||\mathcal{K}|} : \sum_{k \in \mathcal{K}} a_{ij} \, \lambda^k_j \, u^k_{jt} \leq x_{it} \text{ and } \sum_{k \in \mathcal{K}} u^k_{jt} = 1 \quad \forall j \in \mathcal{J} \Big\}.$$

We note that the definition of $\mathcal{U}^i(x_{it})$ is similar to that of $\mathcal{U}(x_t)$ in (1)-(2), but $\mathcal{U}^i(x_{it})$ only imposes the capacity availability on flight leg i.

The optimal total expected revenue in the single leg revenue management problem that takes place over flight leg i is given by $v^i_1(c_i)$. Furthermore, by the discussion above, $\hat{z}^i_{LP}$ provides an upper bound on the optimal total expected revenue in this single leg revenue management problem. This implies that $v^i_1(c_i) \leq \hat{z}^i_{LP}$. If we add over all $i \in \mathcal{L}$ and recall that we have $\sum_{i \in \mathcal{L}} \hat{z}^i_{LP} = \hat{z}_{LP}$, then we obtain $\sum_{i \in \mathcal{L}} v^i_1(c_i) \leq \hat{z}_{LP}$. On the other hand, the next proposition shows that $V_1(c) \leq \sum_{i \in \mathcal{L}} v^i_1(c_i)$ and we obtain $V_1(c) \leq \sum_{i \in \mathcal{L}} v^i_1(c_i) \leq \hat{z}_{LP}$. Therefore, we can solve the optimality equation in (19) to obtain an upper bound on the optimal total expected revenue and this upper bound is tighter than the one provided by the optimal objective value of problem (4)-(7). Solving the optimality equation in (19) is tractable since this optimality equation involves a single dimensional state variable.

Proposition 2 *We have $V_t(x_t) \leq \sum_{i \in \mathcal{L}} v^i_t(x_{it})$ for all $t \in \mathcal{T}$.*

In addition to bounding the optimal total expected revenue, we can use the optimality equation in (19) to make the pricing decisions. In particular, we can approximate the value functions $\{V_t(\cdot) : t \in \mathcal{T}\}$

with separable upper bounds $\{\tilde{V}_t(\cdot) : t \in \mathcal{T}\}$ of the form $\tilde{V}_t(x_t) = \sum_{i \in \mathcal{L}} v_t^i(x_{it})$. In this case, we can replace the value functions $\{V_t(\cdot) : t \in \mathcal{T}\}$ on the right side of problem (3) with the separable value function approximations $\{\tilde{V}_t(\cdot) : t \in \mathcal{T}\}$ and solve this problem to make the pricing decisions at time period t. We refer to this decision rule as DRA, standing for decomposition by revenue allocation. Our choice of terminology is motivated by the fact that $\{\hat{\mu}_{ij}^k : i \in \mathcal{L}\}$ serve as the portions of the expected revenue r_j^k that are allocated to the different flight legs.

4 Decomposition by Leg Relaxation

In this section, we describe a second decision rule that also addresses the stochastic nature of the itinerary requests. Similar to the DRA decision rule in the previous section, the starting point for this decision rule is a duality argument on problem (4)-(7), but the specifics of the duality argument is different. We let $\{\hat{\pi}_i : i \in \mathcal{L}\}$ be the optimal values of the dual variables associated with constraints (5) in problem (4)-(7). We pick an arbitrary flight leg i and relax constraints (5) for all other flight legs by associating the dual multipliers $\{\hat{\pi}_l : l \in \mathcal{L} \setminus \{i\}\}$. In this case, by linear programming duality, problem (4)-(7) has the same optimal objective value as the problem

$$\max \quad \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} \Big[r_j^k - \sum_{l \in \mathcal{L} \setminus \{i\}} a_{lj} \, \lambda_j^k \, \hat{\pi}_l \Big] w_j^k + \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l \, c_l \tag{20}$$

$$\text{subject to} \quad \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} a_{ij} \, \lambda_j^k \, w_j^k \leq c_i \tag{21}$$

$$(6), (7). \tag{22}$$

We recall that we use $\hat{z}_{LP}$ to denote this common optimal objective value. Ignoring the constant term $\sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l \, c_l$ in the objective function above and comparing problem (20)-(22) with problem (4)-(7), we observe that problem (20)-(22) corresponds to the deterministic linear program for a revenue management problem that takes place over the single flight leg i. In this single leg revenue management problem, if we charge the price level k for itinerary j, then the expected revenue that we generate at a time period from itinerary j is given by $r_j^k - \sum_{l \in \mathcal{L} \setminus \{i\}} a_{lj} \, \lambda_j^k \, \hat{\pi}_l$. Since by Proposition 1, the optimal objective value of the deterministic linear program provides an upper bound on the optimal total expected revenue, $\hat{z}_{LP} - \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l \, c_l$ provides an upper bound on the optimal total expected revenue in the single leg revenue management problem that takes place over flight leg i.

We can use an optimality equation similar to the one in (19) to compute the optimal total expected revenue in the single leg revenue management problem that takes place over flight leg i. In particular, this optimality equation is given by

$$\vartheta_t^i(x_{it}) = \max_{u_t \in \mathcal{U}^i(x_{it})} \Bigg\{ \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} u_{jt}^k \Big\{ r_j^k - \sum_{l \in \mathcal{L} \setminus \{i\}} a_{lj} \, \lambda_j^k \, \hat{\pi}_l + \lambda_j^k \, \vartheta_{t+1}^i(x_{it} - a_{ij}) \Big\} + \Big[1 - \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} u_{jt}^k \, \lambda_j^k \Big] \vartheta_{t+1}^i(x_{it}) \Bigg\} \tag{23}$$

with the boundary condition that $\vartheta_{\tau+1}^i(\cdot) = 0$. The optimal total expected revenue in the single leg revenue management problem that takes place over flight leg i is $\vartheta_1^i(c_i)$ and by the discussion in the previous paragraph, we have $\vartheta_1^i(c_i) \leq \hat{z}_{LP} - \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l \, c_l$. On the other hand, the next proposition shows that $V_1(c) \leq \vartheta_1^i(c_i) + \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l \, c_l$ and we obtain $V_1(c) \leq \vartheta_1^i(c_i) + \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l \, c_l \leq \hat{z}_{LP}$. Therefore, we can solve the optimality equation in (23) to obtain an upper bound on the optimal total expected revenue and this upper bound is tighter than the one provided by the optimal objective value of problem (4)-(7). Furthermore, since the choice of flight leg i is completely arbitrary, the last chain of inequalities hold for all $i \in \mathcal{L}$, in which case, we can take the minimum over all $i \in \mathcal{L}$ and use

$$\min_{i \in \mathcal{L}} \left\{ \vartheta_1^i(c_i) + \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l \, c_l \right\}$$

as the tightest possible upper bound on the optimal total expected revenue and this upper bound is also tighter than the one provided by $\hat{z}_{LP}$.

Proposition 3 *We have $V_t(x_t) \leq \vartheta_t^i(x_{it}) + \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l \, x_{lt}$ for all $i \in \mathcal{L}$, $t \in \mathcal{T}$.*

Each of $\{\vartheta_t^i(x_{it}) + \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l \, x_{lt} : i \in \mathcal{L}\}$ provides an upper bound on $V_t(x_t)$, but it is not clear which one of these upper bounds to use as a value function approximation when making the pricing decisions. A natural approach is to average over all $i \in \mathcal{L}$ and make the pricing decisions by using

$$\tilde{V}_t(x_t) = \frac{1}{|\mathcal{L}|} \sum_{i \in \mathcal{L}} \Big\{ \vartheta_t^i(x_{it}) + \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l \, x_{lt} \Big\} \tag{24}$$

as an approximation to $V_t(x_t)$. Unfortunately, this idea does not perform too much better than the DLP-D decision rule. To see the reason, we first observe that if we make the pricing decisions by replacing the value functions $\{V_t(\cdot) : t \in \mathcal{T}\}$ in problem (3) by any value function approximations $\{\tilde{V}_t(\cdot) : t \in \mathcal{T}\}$, then we need to solve a problem of the form

$$\begin{aligned} \max_{u_t \in \mathcal{U}(x_t)} & \left\{ \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} u_{jt}^k \Big\{ r_j^k + \lambda_j^k \, \tilde{V}_{t+1}(x_t - \textstyle\sum_{i \in \mathcal{L}} e_i \, a_{ij}) \Big\} + \Big[1 - \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} u_{jt}^k \, \lambda_j^k \Big] \, \tilde{V}_{t+1}(x_t) \right\} \\ &= \max_{u_t \in \mathcal{U}(x_t)} \left\{ \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} u_{jt}^k \Big\{ r_j^k + \lambda_j^k \, \tilde{V}_{t+1}(x_t - \textstyle\sum_{i \in \mathcal{L}} e_i \, a_{ij}) - \lambda_j^k \, \tilde{V}_{t+1}(x_t) \Big\} \right\} + \tilde{V}_{t+1}(x_t). \end{aligned} \tag{25}$$

Focusing on the second problem above, since the last term $\tilde{V}_{t+1}(x_t)$ is independent of the pricing decisions at time period t, the term that really affects the quality of the pricing decisions is the difference $\tilde{V}_{t+1}(x_t) - \tilde{V}_{t+1}(x_t - \sum_{i \in \mathcal{L}} e_i \, a_{ij})$.

We proceed to compare the form of the difference $\tilde{V}_{t+1}(x_t) - \tilde{V}_{t+1}(x_t - \sum_{i \in \mathcal{L}} e_i \, a_{ij})$ for the value function approximations used by the DLP-D decision rule and for the value function approximations given in (24). As described in Section 2, the value function approximations used by the DLP-D decision rule is of the form $\tilde{V}_t(x_t) = \sum_{i \in \mathcal{L}} \hat{\pi}_i \, x_{it}$ for all $t \in \mathcal{T}$. Therefore, for the value function approximations used by the DLP-D decision rule, we have $\tilde{V}_{t+1}(x_t) - \tilde{V}_{t+1}(x_t - \sum_{i \in \mathcal{L}} e_i \, a_{ij}) = \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\pi}_i$. On the other hand, for the value function approximations given in (24), we have

$$\tilde{V}_{t+1}(x_t) - \tilde{V}_{t+1}(x_t - \textstyle\sum_{i \in \mathcal{L}} a_{ij} \, e_i) = \displaystyle\frac{1}{|\mathcal{L}|} \sum_{i \in \mathcal{L}} \Big\{ \vartheta_{t+1}^i(x_{it}) - \vartheta_{t+1}^i(x_{it} - a_{ij}) + \sum_{l \in \mathcal{L} \setminus \{i\}} a_{lj} \, \hat{\pi}_l \Big\}. \tag{26}$$

We let $\mathcal{L}^j$ be the set of flight legs that are used by itinerary j. If $i \notin \mathcal{L}^j$, then we have $a_{ij} = 0$ by definition so that the sum in the curly brackets above can succinctly be written as $\sum_{l \in \mathcal{L} \setminus \{i\}} a_{lj}\, \hat{\pi}_l = \sum_{l \in \mathcal{L}} a_{lj}\, \hat{\pi}_l$ whenever $i \notin \mathcal{L}^j$. Furthermore, we have $\vartheta^i_{t+1}(x_{it}) - \vartheta^i_{t+1}(x_{it} - a_{ij}) = 0$ for all $i \notin \mathcal{L}^j$. In this case, we can write (26) as

$$\tilde{V}_{t+1}(x_t) - \tilde{V}_{t+1}(x_t - \textstyle\sum_{i \in \mathcal{L}} a_{ij}\, e_i)$$
$$= \frac{1}{|\mathcal{L}|} \Bigg\{ \sum_{i \in \mathcal{L}^j} \Big\{ \vartheta^i_{t+1}(x_{it}) - \vartheta^i_{t+1}(x_{it} - a_{ij}) + \sum_{l \in \mathcal{L} \setminus \{i\}} a_{lj}\, \hat{\pi}_l \Big\} + \sum_{i \in \mathcal{L} \setminus \mathcal{L}^j} \Big[\sum_{l \in \mathcal{L}} a_{lj}\, \hat{\pi}_l \Big] \Bigg\}. \tag{27}$$

One way to visualize the expression on the right side above is that each flight leg contributes one term to the average. A flight leg $i \in \mathcal{L}^j$ contributes the term $\vartheta^i_{t+1}(x_{it}) - \vartheta^i_{t+1}(x_{it} - a_{ij}) + \sum_{l \in \mathcal{L} \setminus \{i\}} a_{lj}\, \hat{\pi}_l$, whereas a flight leg $i \notin \mathcal{L}^j$ contributes the term $\sum_{l \in \mathcal{L}} a_{lj}\, \hat{\pi}_l$. In general, the number of flight legs in the airline network that are not used by itinerary j is much larger than the number of flight legs that are used by itinerary j. This implies that we would expect the average in (27) to be dominated by the terms $\sum_{l \in \mathcal{L}} a_{lj}\, \hat{\pi}_l$ contributed by the flight legs that are not used by itinerary j, in which case, the average in (27) would be very close to $\sum_{l \in \mathcal{L}} a_{lj}\, \hat{\pi}_l$. Therefore, for the value function approximations used by the DLP-D decision rule and for the value function approximations given in (24), the values of the difference $\tilde{V}_{t+1}(x_t) - \tilde{V}_{t+1}(x_t - \sum_{i \in \mathcal{L}} e_i\, a_{ij})$ are very similar and using the value function approximations in (24) does not provide too much improvement over using the DLP-D decision rule.

To deal with this difficulty, instead of taking an average over all flight legs as in (27), we only take an average over the flight legs $i \in \mathcal{L}^j$. In particular, we replace the difference $\tilde{V}_{t+1}(x_t) - \tilde{V}_{t+1}(x_t - \sum_{i \in \mathcal{L}} a_{ij}\, e_i)$ in problem (25) with

$$\frac{1}{|\mathcal{L}^j|} \sum_{i \in \mathcal{L}^j} \Big\{ \vartheta^i_{t+1}(x_{it}) - \vartheta^i_{t+1}(x_{it} - a_{ij}) + \sum_{l \in \mathcal{L} \setminus \{i\}} a_{lj}\, \hat{\pi}_l \Big\}$$

and solve this problem to make the pricing decisions at time period t. We refer to this decision rule as DLR, standing for decomposition by leg relaxation. Our choice of terminology is motivated by the fact that the DLR decision rule is obtained by relaxing the capacity constraints in problem (4)-(7).

Both the DRA and DLR decision rules are obtained by building on problem (4)-(7). Therefore, it is possible to follow the discussion at the end of Section 2 so as to extend the DRA and DLR decision rules to handle the case where the probability of observing a request for itinerary j does not depend only on the price for itinerary j, but also on the prices for the other itineraries.

5 Computational Experiments

In this section, we numerically compare the upper bounds and total expected revenues obtained by the decision rules that we describe in Sections 2, 3 and 4.

5.1 Experimental Setup and Benchmark Strategies

In our computational experiments, we consider two types of functions that capture the relationship between the price and the probability of observing an itinerary request. In the first type of function, we assume that the probability of observing an itinerary request is a linear function of the price. In particular, we let $\Lambda_j(p) = \rho_j\,[1-p/\kappa_j]$ so that if we charge the price p for itinerary j, then the probability of observing a request for itinerary j at a time period is given by $\Lambda_j(p)$. The parameter $\rho_j \in [0,1]$ can be interpreted as the probability of observing a request for itinerary j when we do not charge anything for this itinerary and the parameter $\kappa_j > 0$ can be interpreted as the price sensitivity. The price for itinerary j ranges over the interval $[0, \kappa_j]$ so that we have $\Lambda_j(p) \in [0, \rho_j]$ for all $p \in [0, \kappa_j]$. To work with a finite set of price and probability pairs $\{(p_j^k, \lambda_j^k) : k \in \mathcal{K}\}$, we discretize the interval $[0, \rho_j]$ into 40 equal pieces and focus on the probabilities $\lambda_j^k = (k-1)\rho_j/40$ and the prices $p_j^k = \Lambda_j^{-1}(\lambda_j^k)$ for all $k = 1, \ldots, 41$, where $\Lambda_j^{-1}(\cdot)$ denotes the functional inverse of $\Lambda_j(\cdot)$. We note that if the price for an itinerary ranges roughly over an interval of width \$2,000, then 41 prices provide a price increment of \$50. Such a price increment should be precise enough from practical perspective. In the second type of function, we assume that the probability of observing a request for an itinerary is an exponential function of the price. In particular, we let $\Lambda_j(p) = \rho_j\, e^{-p/\kappa_j}$, where the interpretations for ρ_j and κ_j are the same as in the linear case. We assume that the price for itinerary j ranges over the interval $[0, \ln(10)\,\kappa_j]$ so that we have $\Lambda_j(p) \in [\rho_j/10, \rho_j]$ for all $p \in [0, \ln(10)\,\kappa_j]$. Similar to the linear case, we discretize the interval $[\rho_j/10, \rho_j]$ into 40 equal pieces and focus on the probabilities $\lambda_j^k = [\rho_j/10] + (k-1)\,9\,\rho_j/400$ and the prices $p_j^k = \Lambda_j^{-1}(\lambda_j^k)$ for all $k = 1, \ldots, 41$. In addition to these 41 price and probability pairs, to obtain some $\phi \in \mathcal{K}$ such that $\lambda_j^\phi = 0$, we assume that ∞ is an admissible price and if we charge this price, then the probability of observing an itinerary request is zero.

Our test problems are based on those in Kunnumkal and Topaloglu (2009). We consider an airline network that serves N spokes from a single hub. There is one flight leg from each spoke to the hub and another flight leg from the hub to each spoke. Figure 1 shows the structure of the airline network with $N = 8$. There are two itineraries associated with every possible origin destination pair (o, d) with $o \neq d$. One of these itineraries is highly price sensitive and the other one is moderately price sensitive. Therefore, there are $2N$ flight legs and $2N(N+1)$ itineraries, $4N$ of which include one flight leg and $2N(N-1)$ of which include two flight legs. The price sensitivity associated with a highly price sensitive itinerary is κ times larger than the price sensitivity associated with the corresponding moderately price sensitive itinerary. To measure the tightness of the leg capacities, we let $\hat{k}^j = \mathrm{argmax}_{k \in \mathcal{K}}\{r_j^k\}$ so that the price and probability pair $(p_j^{\hat{k}^j}, \lambda_j^{\hat{k}^j})$ maximizes the one period expected revenue from itinerary j. If we charge the prices $\{p_j^{\hat{k}^j} : j \in \mathcal{J}\}$ for the itineraries, then the total expected demand for the capacity on flight leg i is $\tau \sum_{j \in \mathcal{J}} a_{ij}\, \lambda_j^{\hat{k}^j}$ and we measure the tightness of the leg capacities by

$$\gamma = \frac{\tau \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} a_{ij}\, \lambda_j^{\hat{k}^j}}{\sum_{i \in \mathcal{L}} c_i}.$$

We use (T, N, γ, κ) to label our test problems, where $T \in \{\mathrm{L}, \mathrm{E}\}$ denotes whether $\{\Lambda_j(\cdot) : j \in \mathcal{J}\}$ are linear or exponential functions and the remaining three components are as described above. We vary (T, N, γ, κ) over $\{\mathrm{L}, \mathrm{E}\} \times \{4, 8\} \times \{1.2, 1.6, 2.0\} \times \{2, 4, 8\}$ and this provides 36 test problems. We note that careful network revenue management practices are particularly important when the leg capacities are tight. The values that we use for γ are greater than one, characterizing tight leg capacities.

We use three benchmark strategies. Our first benchmark strategy corresponds to the DLP-P decision rule that we describe at the end of Section 2. Our practical implementation of this decision rule divides the planning horizon into S equal segments and resolves problem (4)-(7) at time periods $\{1+(s-1)\tau/S : s = 1, \ldots, S\}$. In particular, at the beginning of segment s, we replace the right side of constraints (5) with the current remaining leg capacities $\{x_{i,1+(s-1)\tau/S} : i \in \mathcal{L}\}$ and the right side of constraints (6) with the current remaining number of time periods $\tau - (s-1)\tau/S$. We solve problem (4)-(7) and letting $\{\hat{w}_j^k : j \in \mathcal{J},\ k \in \mathcal{K}\}$ be an optimal solution to this problem, we charge the price p_j^k for itinerary j with probability $\hat{w}_j^k/[\tau - (s-1)\tau/S]$ until we reach the beginning of the next segment. A few setup runs indicated that increasing S beyond 12 does not improve the performance of the DLP-P decision rule noticeably so that we use $S = 12$. The performances of the DLP-P and DLP-D decision rules turn out to be virtually identical in all of our test problems and we do not provide detailed results for the DLP-D decision rule. Our second benchmark strategy corresponds to the DRA decision rule that we describe at the end of Section 3, whereas our third benchmark strategy corresponds to the DLR decision rule that we describe at the end of Section 4. For the DRA and DLR decision rules, it is also possible to divide the planning horizon into equal segments and resolve problems (8)-(12) and (4)-(7) at the beginning of each segment to obtain new values for $\{\hat{\mu}_{ij}^k : i \in \mathcal{L},\ j \in \mathcal{J},\ k \in \mathcal{K}\}$ and $\{\hat{\pi}_i : i \in \mathcal{L}\}$, but it turns out that this extension does not provide any noticeable improvement for these decision rules.

5.2 Computational Results

Our main computational results are summarized in Tables 1 and 2. In particular, Table 1 and 2 respectively show the results for the test problems where $\{\Lambda_j(\cdot) : j \in \mathcal{J}\}$ are linear and exponential functions. The first column in these tables shows the problem characteristics. The second, third and fourth columns respectively show the upper bounds on the optimal total expected revenue obtained by DLP-P, DRA and DLR. The fifth and sixth columns show the percent gaps between the upper bounds obtained by DRA and the remaining two benchmark strategies. The upper bounds obtained by DRA are consistently the tightest and we use DRA as a reference when comparing the upper bounds. The seventh, eighth and ninth columns respectively show the total expected revenues obtained by DLP-P, DRA and DLR. We estimate these total expected revenues by simulating the performances of the different benchmark strategies under 100 sample paths. We use common random numbers when simulating the performances of the different benchmark strategies. The tenth and eleventh columns show the percent gaps between the total expected revenues obtained by DLR and the remaining two benchmark strategies. The total expected revenues obtained DLR are consistently the highest and we use DLR as a reference when comparing the total expected revenues. For all of our test problems, we carried out paired t-tests to compare the total expected revenues obtained by DLR with those obtained by DLP-P and DRA. These tests indicated that the performance gaps between DLR and the other two benchmark strategies are statistically significant at 5% level for all of our test problems. We note that our paired t-tests compare the total expected revenues obtained by DLR with those obtained DLP-P and DRA. We do not compare the total expected revenues obtained by DLP-P and DRA with each other and the performance gaps between DLP-P and DRA may or may not be statistically significant.

Comparing the upper bounds obtained by the three benchmark strategies, we observe that the upper bounds obtained by DRA are significantly tighter than those obtained by DLP-P and DLR. For the test problems in Table 1, the average gap between the upper bounds obtained by DRA and DLP-P is 3.28%, whereas the average gap between the upper bounds obtained by DRA and DLR is 2.04%. When we move to the test problems in Table 2, the upper bounds obtained by DRA improve those obtained by DLP-P and DLR by respectively 3.94% and 2.57% on average. There are test problems where the gap between the upper bounds obtained by DRA and DLP-P is as high as 7.98% and the gap between the upper bounds obtained by DRA and DLR is as high as 6.11%. In all of our test problems, the upper bounds obtained by DRA are uniformly tighter than those obtained by DLP-P and DLR.

Comparing the total expected revenues obtained by the three benchmark strategies, we observe that DLR obtains significantly higher total expected revenues than DLP-P and DRA. For the test problems in Table 1, the average gap between the total expected revenues obtained by DLR and DLP-P is 5.05%, whereas the average gap between the total expected revenues obtained by DLP and DRA is 2.94%. When we move to the test problems in Table 2, the total expected revenues obtained by DLR improve those obtained by DLP-P and DRA by respectively 9.17% and 3.41% on average. There are test problems where the gap between the total expected revenues obtained by DLR and DLP-P is as high as 17.02% and the gap between the total expected revenues obtained by DLR and DRA is as high as 5.53%. In all of our test problems, the total expected revenues obtained by DLR are uniformly higher than those obtained by DLP-P and DRA. Although the total expected revenues obtained by DRA are not as high as those obtained by DLR, DRA can provide noticeable improvements over DLP-P.

DLP-P represents one of the traditional approaches for solving pricing problems and our results indicate that DRA and DLR complement each other as they provide improvements over DLP-P. In particular, DRA tightens the upper bounds and allows us to assess the optimality gaps more accurately, whereas DLR obtains higher total expected revenues. To give a feel for the problem parameters that affect the relative performances of the three benchmark strategies, Figure 2 plots the gaps between the upper bounds obtained by DRA and the remaining two benchmark strategies and Figure 3 plots the gaps between the total expected revenues obtained by DLR and the remaining two benchmark strategies. The test problems in the horizontal axis in these figures are arranged in such a fashion that the first and last nine test problems respectively involve four and eight spokes. The leg capacities get tighter as we move from left to right within a block of nine test problems. The difference between the price sensitivities of the highly and moderately price sensitive itineraries gets larger as we move from left to right within a block of three test problems. For economy of space, Figures 2 and 3 consider only the case where $\{\Lambda_j(\cdot) : j \in \mathcal{J}\}$ are exponential functions. Figure 2 indicates that the gaps between the upper bounds obtained by DRA and the remaining two benchmark strategies get larger as the leg capacities get tighter, whereas Figure 3 indicates that the gaps between the total expected revenues obtained by DLR and the remaining two benchmark strategies get larger as the difference in the price sensitivities gets larger. If we had infinite capacity on the flight legs, then the different time periods in the planning horizon would not interact. In this case, letting $\hat{k}^j = \mathrm{argmax}_{k \in \mathcal{K}} \{r_j^k\}$, it would be trivially optimal to charge the prices $\{p_j^{\hat{k}^j} : j \in \mathcal{J}\}$ for the itineraries. Therefore, we intuitively expect the test problems with tight leg capacities to be more difficult. On the other hand, if the difference in the price sensitivities of the highly and moderately price sensitive itineraries gets larger, then we use a richer set of prices to obtain good performance. Therefore, we also intuitively expect the test problems with larger differences in the price sensitivities to be more difficult. These observations indicate that DRA obtains especially tighter upper bounds and DLR obtains especially higher total expected revenues for the test problems that we intuitively expect to be more difficult.

Table 3 shows the CPU seconds for DRA and DLR with different numbers of time periods in the planning horizon and with different numbers of spokes in the airline network. All of our computational experiments are carried out on a Pentium IV PC running Windows XP with 2.4 Ghz CPU and 1 GB RAM. The CPU seconds for DRA correspond to the time required to solve problem (8)-(12) and the optimality equation in (19), whereas the CPU seconds for DLR correspond to the time required to solve problem (4)-(7) and the optimality equation in (23). The results indicate that the CPU seconds for DRA are noticeably longer than those for DLR. This discrepancy is due to the fact that DRA is based on problem (8)-(12) and this problem is significantly larger than problem (4)-(7), which forms the starting point for DLR. Even for the largest test problems, the CPU seconds for both DRA and DLR are quite reasonable. The CPU seconds for DLP-P are on the order of a fraction of a second and we do not provide detailed CPU seconds for DLP-P. Overall, considering their improvements over DLP-P in terms of both upper bounds and total expected revenues, DRA and DLR are strong candidates for solving practical pricing problems.

6 Conclusions

In this paper, we developed two methods for making pricing decisions in network revenue management problems. Both methods decompose the dynamic programming formulation of the problem by the flight legs and solve dynamic programs with single dimensional state variables. We established that our methods obtain upper bounds on the optimal total expected revenue and these upper bounds are tighter than the one obtained by the deterministic linear program. Our computational experiments demonstrated significant improvements over the deterministic linear program and indicated that the two methods complement each other. In particular, the first method is useful in obtaining tight bounds on the optimal total expected revenue, whereas the pricing policy from the second method obtains higher total expected revenues.

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Appendix: Omitted Proofs

Proof of Proposition 1 We let $\{\hat{U}^k_{jt} : j \in \mathcal{J},\ k \in \mathcal{K},\ t \in \mathcal{T}\}$ be the pricing decisions under the optimal control policy, where $\hat{U}^k_{jt} = 1$ if we charge the price p^k_j for itinerary j at time period t and $\hat{U}^k_{jt} = 0$ otherwise. Similarly, we let $\hat{S}_{jt} = 1$ if we serve a request for itinerary j at time period t under the optimal control policy and $\hat{S}_{jt} = 0$ otherwise. We note that $\{\hat{U}^k_{jt} : j \in \mathcal{J},\ k \in \mathcal{K},\ t \in \mathcal{T}\}$ and $\{\hat{S}_{jt} : j \in \mathcal{J},\ t \in \mathcal{T}\}$ are random variables and we have $\sum_{k\in\mathcal{K}} \hat{U}^k_{jt} = 1$ for all $j \in \mathcal{J}$, $t \in \mathcal{T}$ by the feasibility of the pricing decisions. Furthermore, using $\hat{U}_{jt}$ to denote the vector $\{\hat{U}^k_{jt} : k \in \mathcal{K}\}$, we have $\mathbb{E}\{\hat{S}_{jt}\} = \mathbb{E}\{\mathbb{E}\{\hat{S}_{jt} \,|\, \hat{U}_{jt}\}\} = \sum_{k\in\mathcal{K}} \mathbb{E}\{\hat{S}_{jt} \,|\, \hat{U}^k_{jt} = 1\}\, \mathbb{P}\{\hat{U}^k_{jt} = 1\} = \sum_{k\in\mathcal{K}} \lambda^k_j\, \mathbb{E}\{\hat{U}^k_{jt}\}$, where the last equality follows from the fact that $\mathbb{E}\{\hat{S}_{jt} \,|\, \hat{U}^k_{jt} = 1\} = \lambda^k_j$ and $\mathbb{P}\{\hat{U}^k_{jt} = 1\} = \mathbb{E}\{\hat{U}^k_{jt}\}$ since $\hat{U}^k_{jt}$ is a Bernoulli random variable. Under the optimal control policy, the price that we charge for itinerary j at time period t is $\sum_{k\in\mathcal{K}} p^k_j\, \hat{U}^k_{jt}$. Thus, letting $\hat{\Pi}$ be the optimal total expected revenue, we have

$$\begin{aligned}\hat{\Pi} = \mathbb{E}\bigg\{ \sum_{t\in\mathcal{T}} \sum_{j\in\mathcal{J}} \hat{S}_{jt} \Big[\sum_{k\in\mathcal{K}} p^k_j\, \hat{U}^k_{jt} \Big] \bigg\} &= \sum_{t\in\mathcal{T}} \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} p^k_j\, \mathbb{E}\{\hat{S}_{jt}\, \hat{U}^k_{jt}\} \\ &= \sum_{t\in\mathcal{T}} \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} p^k_j\, \mathbb{E}\{\hat{S}_{jt} \,|\, \hat{U}^k_{jt} = 1\}\, \mathbb{P}\{\hat{U}^k_{jt} = 1\} = \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} p^k_j\, \lambda^k_j \Big[\sum_{t\in\mathcal{T}} \mathbb{E}\{\hat{U}^k_{jt}\} \Big]. \end{aligned} \tag{28}$$

On the other hand, since the itinerary requests that we serve satisfy the capacity constraints, we have $\sum_{t\in\mathcal{T}} \sum_{j\in\mathcal{J}} a_{ij}\, \hat{S}_{jt} \le c_i$ for all $i \in \mathcal{L}$. Taking expectations in the last inequality and noting that $\mathbb{E}\{\hat{S}_{jt}\} = \sum_{k\in\mathcal{K}} \lambda^k_j\, \mathbb{E}\{\hat{U}^k_{jt}\}$, we obtain

$$\sum_{t\in\mathcal{T}} \sum_{j\in\mathcal{J}} a_{ij}\, \mathbb{E}\{\hat{S}_{jt}\} = \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} a_{ij}\, \lambda^k_j \Big[\sum_{t\in\mathcal{T}} \mathbb{E}\{\hat{U}^k_{jt}\} \Big] \le c_i. \tag{29}$$

By the feasibility of the pricing decisions, we have $\sum_{k\in\mathcal{K}} \hat{U}^k_{jt} = 1$ for all $j \in \mathcal{J}$, $t \in \mathcal{T}$. If we take expectations and add over all time periods, then we obtain

$$\sum_{k\in\mathcal{K}} \Big[\sum_{t\in\mathcal{T}} \mathbb{E}\{\hat{U}^k_{jt}\} \Big] = \tau. \tag{30}$$

By (29) and (30), $\{\sum_{t\in\mathcal{T}} \mathbb{E}\{\hat{U}^k_{jt}\} : j \in \mathcal{J},\ k \in \mathcal{K}\}$ is a feasible solution to problem (4)-(7). Furthermore, the objective value provided by this feasible solution is $\sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} r^k_j\, [\sum_{t\in\mathcal{T}} \mathbb{E}\{\hat{U}^k_{jt}\}] = \hat{\Pi}$, where the equality follows by (28). Thus, the optimal objective value of problem (4)-(7) is at least $\hat{\Pi}$. □

Proof of Proposition 2 We show the result by induction over the time periods. For time period $\tau + 1$, we have $V_{\tau+1}(\cdot) = 0$ and $v^i_{\tau+1}(\cdot) = 0$ for all $i \in \mathcal{L}$ so that the result holds trivially for time period $\tau + 1$. Assuming that the result holds for time period $t + 1$ and letting $\hat{u}_t$ be the optimal solution to problem (3), we have

$$\begin{aligned} V_t(x_t) &= \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} \hat{u}^k_{jt} \Big\{ r^k_j + \lambda^k_j\, V_{t+1}(x_t - \textstyle\sum_{i\in\mathcal{L}} e_i\, a_{ij}) \Big\} + \Big[1 - \displaystyle\sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} \hat{u}^k_{jt}\, \lambda^k_j \Big] V_{t+1}(x_t) \\ &\le \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} \hat{u}^k_{jt} \Big\{ r^k_j + \lambda^k_j \sum_{i\in\mathcal{L}} v^i_{t+1}(x_{it} - a_{ij}) \Big\} + \Big[1 - \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} \hat{u}^k_{jt}\, \lambda^k_j \Big] \sum_{i\in\mathcal{L}} v^i_{t+1}(x_{it}) \\ &= \sum_{i\in\mathcal{L}} \bigg\{ \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} \hat{u}^k_{jt} \Big\{ \hat{\mu}^k_{ij} + \lambda^k_j\, v^i_{t+1}(x_{it} - a_{ij}) \Big\} + \Big[1 - \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} \hat{u}^k_{jt}\, \lambda^k_j \Big] v^i_{t+1}(x_{it}) \bigg\} \le \sum_{i\in\mathcal{L}} v^i_t(x_{it}), \end{aligned}$$

where the first inequality follows from the induction assumption, the second equality follows from the fact that $r_j^k = \sum_{i \in \mathcal{L}} \hat{\mu}_{ij}^k$ for all $j \in \mathcal{J}$, $k \in \mathcal{K}$ and the last inequality follows from the fact that $\hat{u}_t$ is a feasible but not necessarily an optimal solution to problem (19). □

Proof of Proposition 3 We use an induction argument that is similar to the proof of Proposition 2. For time period $\tau + 1$, we have $V_{\tau+1}(\cdot) = 0$, $\vartheta^i_{\tau+1}(\cdot) = 0$ for all $i \in \mathcal{L}$ and $\hat{\pi}_l \geq 0$ for all $l \in \mathcal{L}$ by dual feasibility to problem (4)-(7). Therefore, the result holds for time period $\tau + 1$. Assuming that the result holds for time period $t + 1$ and letting $\hat{u}_t$ be the optimal solution to problem (3), we have

$$
\begin{aligned}
V_t(x_t) &= \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} \hat{u}_{jt}^k \Big\{ r_j^k + \lambda_j^k \, V_{t+1}(x_t - \textstyle\sum_{i \in \mathcal{L}} e_i \, a_{ij}) \Big\} + \Big[1 - \displaystyle\sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} \hat{u}_{jt}^k \, \lambda_j^k \Big] \, V_{t+1}(x_t) \\
&\leq \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} \hat{u}_{jt}^k \Big\{ r_j^k + \lambda_j^k \, \vartheta^i_{t+1}(x_{it} - a_{ij}) + \lambda_j^k \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l \, [x_{lt} - a_{lj}] \Big\} \\
&\qquad + \Big[1 - \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} \hat{u}_{jt}^k \, \lambda_j^k \Big] \Big[\vartheta^i_{t+1}(x_{it}) + \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l \, x_{lt} \Big] \\
&= \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} \hat{u}_{jt}^k \Big\{ r_j^k - \sum_{l \in \mathcal{L} \setminus \{i\}} a_{lj} \, \lambda_j^k \, \hat{\pi}_l + \lambda_j^k \, \vartheta^i_{t+1}(x_{it} - a_{ij}) \Big\} \\
&\qquad + \Big[1 - \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} \hat{u}_{jt}^k \, \lambda_j^k \Big] \, \vartheta^i_{t+1}(x_{it}) + \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l \, x_{lt} \\
&\leq \sum_{i \in \mathcal{L}} \vartheta^i_t(x_{it}) + \sum_{l \in \mathcal{L} \setminus \{i\}} \hat{\pi}_l \, x_{lt},
\end{aligned}
$$

where the first inequality follows from the induction assumption and the last inequality follows from the fact that $\hat{u}_t$ is a feasible but not necessarily an optimal solution to problem (23). □

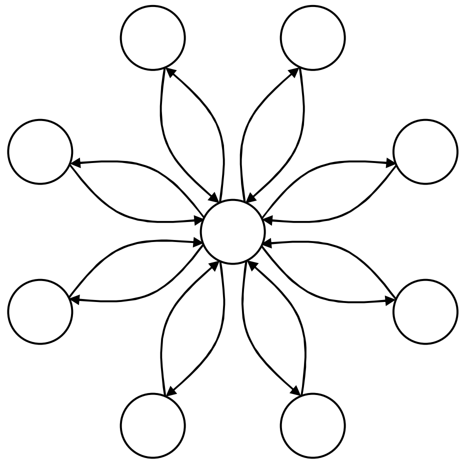

Figure 1: Airline network with eight spokes.

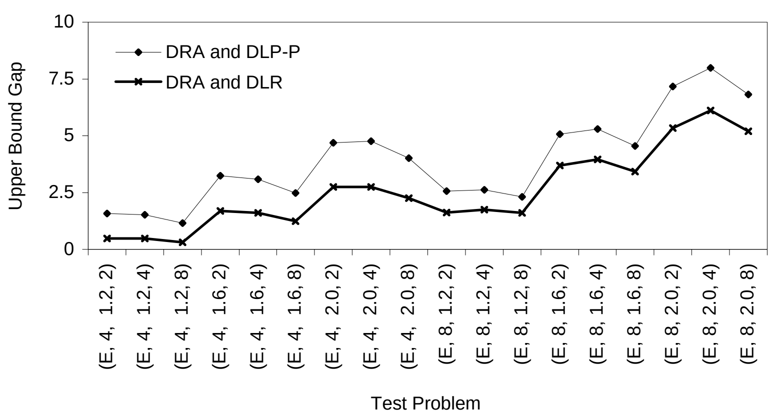

Figure 2: Percent gaps between the upper bounds obtained by DRA and the remaining two benchmark strategies for the test problems where $\{\Lambda_j(\cdot) : j \in \mathcal{J}\}$ are exponential functions.

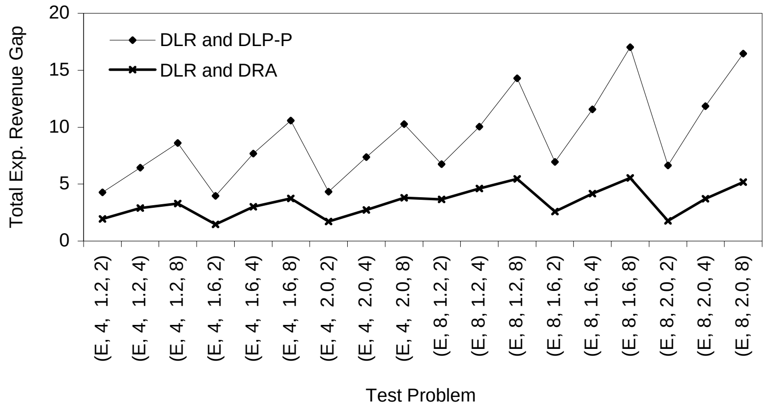

Figure 3: Percent gaps between the total expected revenues obtained by DLR and the remaining two benchmark strategies for the test problems where $\{\Lambda_j(\cdot) : j \in \mathcal{J}\}$ are exponential functions.

Problem	Upper Bound on Optimal Total Expected Revenue			% Gap with DRA		Total Expected Revenue Estimated by Simulation			% Gap with DLR	
(T, N, γ, κ)	DLP-P	DRA	DLR	DLP-P	DLR	DLP-P	DRA	DLR	DLP-P	DRA
$(\mathrm{L}, 4, 1.2, 2)$	6,606	6,495	6,531	1.72	0.57	6,049	6,060	6,189	2.27	2.10
$(\mathrm{L}, 4, 1.2, 4)$	9,538	9,388	9,435	1.59	0.50	8,654	8,729	9,017	4.03	3.20
$(\mathrm{L}, 4, 1.2, 8)$	15,416	15,236	15,286	1.18	0.33	13,938	14,233	14,776	5.68	3.68
$(\mathrm{L}, 4, 1.6, 2)$	5,922	5,760	5,835	2.80	1.30	5,304	5,324	5,396	1.70	1.35
$(\mathrm{L}, 4, 1.6, 4)$	8,792	8,548	8,669	2.86	1.42	7,789	7,905	8,087	3.68	2.25
$(\mathrm{L}, 4, 1.6, 8)$	14,640	14,306	14,474	2.33	1.18	12,963	13,354	13,766	5.83	2.99
$(\mathrm{L}, 4, 2.0, 2)$	5,271	5,098	5,186	3.39	1.72	4,674	4,673	4,738	1.35	1.37
$(\mathrm{L}, 4, 2.0, 4)$	8,084	7,803	7,956	3.61	1.97	7,067	7,171	7,337	3.69	2.27
$(\mathrm{L}, 4, 2.0, 8)$	13,897	13,511	13,724	2.86	1.58	12,144	12,618	12,949	6.22	2.56
$(\mathrm{L}, 8, 1.2, 2)$	6,273	6,108	6,209	2.70	1.65	5,454	5,501	5,655	3.57	2.72
$(\mathrm{L}, 8, 1.2, 4)$	8,924	8,697	8,839	2.61	1.63	7,587	7,768	8,087	6.19	3.95
$(\mathrm{L}, 8, 1.2, 8)$	14,239	13,961	14,132	1.99	1.23	11,951	12,429	13,118	8.90	5.25
$(\mathrm{L}, 8, 1.6, 2)$	5,680	5,431	5,607	4.57	3.24	4,774	4,822	4,930	3.17	2.19
$(\mathrm{L}, 8, 1.6, 4)$	8,281	7,894	8,179	4.90	3.60	6,800	6,975	7,226	5.90	3.47
$(\mathrm{L}, 8, 1.6, 8)$	13,569	13,056	13,434	3.93	2.90	10,959	11,521	12,119	9.57	4.93
$(\mathrm{L}, 8, 2.0, 2)$	5,152	4,897	5,079	5.22	3.73	4,261	4,309	4,396	3.08	1.98
$(\mathrm{L}, 8, 2.0, 4)$	7,710	7,282	7,602	5.87	4.40	6,176	6,412	6,587	6.24	2.66
$(\mathrm{L}, 8, 2.0, 8)$	12,972	12,368	12,825	4.88	3.70	10,241	10,917	11,367	9.91	3.96
Average				3.28	2.04				5.05	2.94

Table 1: Performances of DLP-P, DRA and DLR for the test problems where $\{\Lambda_j(\cdot) : j \in \mathcal{J}\}$ are linear functions.

Problem (T, N, γ, κ)	Upper Bound on Optimal Total Expected Revenue DLP-P	DRA	DLR	% Gap with DRA DLP-P	DLR	Total Expected Revenue Estimated by Simulation DLP-P	DRA	DLR	% Gap with DLR DLP-P	DRA
$(\mathrm{E}, 4, 1.2, 2)$	4,271	4,205	4,225	1.58	0.48	3,899	3,994	4,073	4.26	1.95
$(\mathrm{E}, 4, 1.2, 4)$	6,149	6,057	6,086	1.52	0.48	5,506	5,715	5,886	6.45	2.91
$(\mathrm{E}, 4, 1.2, 8)$	9,909	9,795	9,826	1.15	0.31	8,742	9,251	9,567	8.62	3.30
$(\mathrm{E}, 4, 1.6, 2)$	4,018	3,892	3,958	3.23	1.69	3,551	3,643	3,697	3.97	1.46
$(\mathrm{E}, 4, 1.6, 4)$	5,866	5,691	5,782	3.08	1.61	5,067	5,322	5,488	7.67	3.01
$(\mathrm{E}, 4, 1.6, 8)$	9,608	9,375	9,491	2.48	1.24	8,138	8,760	9,101	10.58	3.75
$(\mathrm{E}, 4, 2.0, 2)$	3,723	3,556	3,654	4.69	2.75	3,209	3,297	3,354	4.34	1.72
$(\mathrm{E}, 4, 2.0, 4)$	5,531	5,280	5,425	4.75	2.74	4,614	4,845	4,981	7.37	2.72
$(\mathrm{E}, 4, 2.0, 8)$	9,242	8,885	9,085	4.01	2.25	7,476	8,016	8,331	10.26	3.78
$(\mathrm{E}, 8, 1.2, 2)$	4,050	3,949	4,013	2.56	1.62	3,456	3,571	3,707	6.76	3.67
$(\mathrm{E}, 8, 1.2, 4)$	5,748	5,601	5,699	2.62	1.75	4,751	5,038	5,282	10.05	4.62
$(\mathrm{E}, 8, 1.2, 8)$	9,147	8,941	9,084	2.30	1.60	7,305	8,058	8,523	14.29	5.46
$(\mathrm{E}, 8, 1.6, 2)$	3,827	3,643	3,777	5.06	3.68	3,070	3,213	3,299	6.93	2.60
$(\mathrm{E}, 8, 1.6, 4)$	5,501	5,224	5,431	5.30	3.96	4,192	4,542	4,740	11.56	4.17
$(\mathrm{E}, 8, 1.6, 8)$	8,884	8,498	8,790	4.54	3.43	6,466	7,361	7,792	17.02	5.53
$(\mathrm{E}, 8, 2.0, 2)$	3,558	3,320	3,498	7.17	5.34	2,719	2,860	2,912	6.64	1.78
$(\mathrm{E}, 8, 2.0, 4)$	5,194	4,810	5,104	7.98	6.11	3,686	4,026	4,181	11.85	3.71
$(\mathrm{E}, 8, 2.0, 8)$	8,547	8,002	8,417	6.81	5.19	5,747	6,522	6,878	16.45	5.17
Average				3.94	2.57				9.17	3.41

Table 2: Performances of DLP-P, DRA and DLR for the test problems where $\{\Lambda_j(\cdot) : j \in \mathcal{J}\}$ are exponential functions.

No. Time Pers. (τ)	CPU Seconds DRA	DLR
180	3.43	0.44
360	4.17	1.34
720	7.67	5.02
1,440	22.16	19.52

No. Spokes (N)	CPU Seconds DRA	DLR
4	0.68	0.58
6	1.81	1.06
8	4.17	1.34
10	9.05	1.74

Table 3: CPU seconds for DRA and DLR.