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Convergence of Decreasing Sequences of Convex Sets in Nonreflexive Banach Spaces

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Abstract. We consider nested sequences of linear or convex closed sets of the form arising in estimation and other inverse problems. We show that such sequences may fail to converge in any of the recently studied set convergences other than Mosco convergence. We also provide a positive result concerning the epislice convergence of related sequences of functions.

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1. Introduction

In this note, principally we give some examples showing the limitations of the slice topology and other set topologies on convex (linear) subsets of a nonreflexive Banach space. In many applied optimization or estimation problems (inverse problems), the constraint sets take the form of a nested sequence of closed convex or affine sets. In the reflexive setting, these problems may be addressed using Mosco convergence and related ideas (see [4, 7]). It is well known [4] that Mosco convergence has many defects outside of reflexive spaces and this stimulated the study of other convergences, especially slice-convergence ([1, 2, 3] among many references).

Suppose that X is a Banach space. We recall that a sequence of closed convex sets $\{C_n\}$ is *slice convergent* to C_∞ if, for each closed bounded convex set W, the distance,

$$\operatorname{dist}(C_n, W) \to \operatorname{dist}(C_\infty, W) \text{ as } n \to \infty .$$

Here, the distance is given by

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$$\mathrm{dist}(C, W) := \inf \|c - w\| \ : \ c \in C,\ w \in W\} \ .$$

Less arduously, $\{C_n\}$ is *Wijsman convergent* to C_∞ if, for each point x, the distance, $d_C(x) := \mathrm{dist}(C, x)$ satisfies

$$\mathrm{dist}(C_n, x) \to \mathrm{dist}(C_\infty, x) \ \text{ as } \ n \to \infty \ .$$

We also recall that $\{C_n\}$ is *Mosco convergent* to C_∞, if

$$\text{weak-lim sup } C_n = \text{strong-lim inf } C_n = C_\infty$$

and less arduously $\{C_n\}$ is *Painlevé–Kuratowski convergence* to C_∞ if

$$\text{strong-lim sup } C_n = \text{strong-lim inf } C_n = C_\infty,$$

and refer to [1, 2, 3, 4] for further details.

2. Results

Unhappily, as we proceed to illustrate, outside of reflexive spaces these convergences are largely incompatible with the study of nested (linear) sequences – although clearly monotone decreasing sequences do converge Mosco. We are, however, able to provide a partial redress in Theorem 4 below. For nested sequences of closed convex sets, this incompatibility was already observed in [2] (in the proof of Theorem 3.8 (3) $\Rightarrow$ (1)).

THEOREM 1. *Suppose that X is a Banach space in which every decreasing sequence of closed linear (so convex) sets $\{C_n\}$ with intersection C_∞ converges in the slice topology to C_∞. Then X is reflexive.*

We begin with the following Lemma:

LEMMA 2. *In every nonreflexive separable Banach space X there exists a w^*-dense sequence $\{a_n^*\}$ in X^* is not norm-dense.*

Proof. Let $F \in X^{**} \backslash X$ so that

$$N(F) := \{x^* \in X^* \ : \ \langle F, x^* \rangle = 0\}$$

is w^*-dense in X^* and norm-closed but not w^*-closed. Since X is separable $B(X^*)$ is w^*-metrizable and separable. Thus, so is $B(X^*) \cap N(F)$. If $\{a_n^*\}$ is chosen w^*-dense in $N(F)$, then it is necessarily w^*-dense in X^* but it has norm-closure lying in $N(F)$. □

Proof of Theorem 1. It suffices to assume that X is separable and to build our example therein. Using Lemma 2, let us suppose that

$$\mathrm{span}\{a_n^*,\ n = 1, 2, \ldots\} \tag{$*$}$$

is w^*-dense in X^* but not norm-dense. Set $Y := \overline{\text{span}}\{a_n^*,\ n = 1, 2, \ldots\} \neq X^*$. Let

$$C_n := \{x \in X \ :\ \langle a_k^*, x\rangle = 0 \ \text{ for } \ 1 \le k \le n\}\,.$$

Then $\cap C_n = \{0\} = C_\infty$ and so $C_n \to C_\infty$ Mosco.

We next show that C_n does not converge in slice topology to C_∞. Let $a^* \in S(X^*)\backslash Y$ and pick γ such that $0 < \gamma < d_Y(a^*)$. Let $S := \{x \ :\ \|x\| \le 1,\ \langle a^*, x\rangle \ge \gamma\}$.

We claim that $C_n \cap S \neq \emptyset$ [while clearly $\text{dist}(\{0\}, S) \ge \gamma > 0$]. Otherwise, the value of

$$p := \sup\{\langle a^*, x\rangle \ :\ \langle a_k^*, x\rangle = 0 \ \text{ for } \ 1 \le k \le n\,,\ \ \|x\| \le 1\} \le \gamma\,.$$

By standard arguments (or by the quasi-interior (CQ) [6]), there is a Lagrange multiplier $(\lambda_1, \ldots, \lambda_m)$ such that

$$\left\| a^* - \sum_{k=1}^{n} \lambda_k a_k^* \right\|_* = \sup_{\|x\| \le 1} \ \langle a^* - \sum_{k=1}^{n} \lambda_k a_k^*, x\rangle = p$$

and so $\gamma < d_Y(a^*) \le p \le \gamma$. This contradiction shows that the gap between S and C_n is zero. Hence slice convergence does not occur. □

We are particularly interested in moment problems of the form studied in [5] and [6]. These give rise to constraints of the form $\langle a_k^*, x\rangle = \langle a_k^*, \bar{x}\rangle$ for $1 \le k \le n$; especially in $L_1([a, b], d\lambda)$ where $\{a_k^*\}$ are Hausdorff or Fourier coefficients $[a_k^* = t^{k-1}$ or $a_k^* = e^{ikt}]$.

COROLLARY 3. *If X^* is not separable and $\{a_k^*\}$ is w^*-densely spanning in X^*, set*

$$C_n := \{x \in X \ :\ \langle a_k^*, x\rangle = \langle a_k^*, \bar{x}\rangle \ \text{ for } \ 1 \le k \le n\}\,.$$

Then $\cap C_n = \{\bar{x}\} = C_\infty$ and $C_n \downarrow C_\infty$ Mosco, while C_n does not converge in slice topology to C_∞. In particular, this holds if $X = L^1([a, b], d\lambda)$ and $\{a_n^*\}$ are Hausdorff or Fourier coefficients. Similarly, this is the case for multi-dimensional Hausdorff or Fourier coefficients in $L^1(T, d\lambda)$ (T infinite compact in R^n and $\mu \approx d\lambda$ Lebesgue) or in $C(T)$.

As a positive result we have the following where *epi-slice convergence* denotes slice-convergence of epigraphs:

THEOREM 4. *Suppose that $\{C_n\}$ is a sequence of closed convex sets in a Banach space X such that*

$$(d - M) \quad \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} C_n \supset C_\infty \supset w - \lim_n \, \sup C_n$$

as holds if the sequence is monotone decreasing. Suppose also that $f\ :\ X \to]-\infty,\infty]$ *is convex with an everywhere Mackey continuous Fenchel conjugate (equivalently every linear perturbation of* f *is weakly inf-compact* [7]). *Then* $f_n := f + \delta_{C_n} \to f_\infty := f + \delta_{C_\infty}$ *in the epi-slice topology.*

Proof. From [2] (as improved by Attouch–Beer in the *Sem. Anal. Convex Montpelier*, 1991), we must show (i) and (ii) below.

(i) Given any x_∞ in X there exists $x_n \to x_\infty$ so that $f_n(x_n) \to f_\infty(x_\infty)$.
(ii) Given any y_∞ in X^*, there exists $y_n \to y_\infty$ so that $f_n^*(y_n) \to f_\infty^*(y_\infty)$.

(i) We set $x_n := x_\infty$ for all n. If $x_\infty \in \mathrm{dom}(f_\infty) = C_\infty$, we have $x_\infty \in C_n$ for large n by the discrete liminf inclusion, and so $f_n(x_\infty) = f_\infty(x_\infty)$ for large n. If $x_\infty \notin \mathrm{dom}(f_\infty)$, then $x_\infty \notin C_\infty$ and so $x_\infty \in X \backslash C_n$ for large n as otherwise we violate the weak-limsup inclusion. Thus again, $f_n(x_\infty) = f_\infty(x_\infty) = (= \infty)$.

(ii) We set $y_n := y_\infty$ for all n. Let $g := x \to f(x) - \langle y_\infty, x\rangle$. Then $f_n^*(y_\infty) = -\inf_{x\in C_n}\ g(x)$ for $n = 1\ldots\infty$ and g has weakly compact lower-level sets. By [6, Thm 2.9(ii)] $f_n^*(y_\infty) \to f_\infty^*(y_\infty)$ as required. □

3. Examples and Remarks

Remarks 5. (a) The condition on f is satisfied if $X = L^1(T,\mu)$ with μ a finite measure and

$$f(x) := \int_T \varphi(x(t))\,\mu(\mathrm{d}t) \ \text{ where } \varphi\ :\ R \to]-\infty,\infty]$$

is closed convex with φ^* everywhere finite [6].

(b) By contrast, if $f := 0$ and C_n are as in Theorem 1 or Corollary 3, then $f_n := f + \delta_{C_n}$ does not converge to $f_\infty := f + \delta_{C_\infty}$ in the epi-slice topology. This follows either directly on observing that slice convergence of C_n to C_∞ coincides with epi-slice convergence of δ_{C_n} to δ_{C_∞} or by checking that (ii) fails in the previous argument for any $y_\infty \notin \overline{\mathrm{span}}(a_n^*,\ n = 1,2,\ldots\}$. Indeed $f_\infty^*(y_\infty) = \delta_{C_\infty}^*(y_\infty) = 0$, while if $y_n \to y_\infty$ then eventually $y_n \notin \overline{\mathrm{span}}(a_n^*,\ n = 1,2,\ldots\}$ and so $f_n^*(y_n) = \delta_{C_n}^*(y_n) = \infty$ for n large.

(c) In fact, we have established conditions so that $f_n \to f$ and $f_n^* \to f^*$ pointwise and, a fortiori, in the epi-slice topology.

(d) Even in $I\!R^n$, simple examples show that we may not replace $(d - M)$ by Mosco convergence in the hypotheses of Theorem 4 without a more stringent constraint qualification [7].

(e) The construction of Theorem 1 gives another proof that slice and Mosco convergence coincide for all decreasing sequences of convex (indeed linear) sets if and only if X is reflexive. (See also [2].)

(f) Similarly, Mosco and Painlevé–Kuratowski convergence coincide for all decreasing convex sequences if and only if X is a (possibly finite-dimensional) Schur space (i.e. weak and norm sequential convergence coincide). Indeed, if X

is not Schur, we construct a sequence $\{c_n\}$ with nonzero weak limit c_∞ such that $\inf_n \|c_n - c_\infty\| > 0$. Then set $C_n := \text{co}\{0, c_n\}$ and we check that C_n converges Painlevé–Kuratowski to $\{0\}$ but is not Mosco convergent.

(g) We observe that in any nonreflexive space we may easily construct nested convex (nonlinear) closed bounded subsets $\{C_n\}$ with nonempty intersection C_∞ which do not converge in the sense of Wisjman (point-wise convergence of the respective distance functions in the original norm) and so, a fortiori, do not converge in the slice topology. (Another construction is given in [2], Theorem 3.8.) To see this, let $\bar{x}$ be of norm one and consider $D := \{x \ : \ \|x - \bar{x}\| \leq \frac{1}{2}\}$. As D contains a ball in a nonreflexive space, James theorem assures that we may find a decreasing sequence of closed convex sets $\{D_n\}$ in D with void intersection. If $C_n := \text{co}\{0, D_n\}$ and $C_\infty := \{0\}$ then C_n decreases to $\{0\}$, and converges Mosco, but fails to converge in the sense of Wijsman.

Indeed, $d_{C_n}(\bar{x}) \leq d_{D_n}(\bar{x}) \leq \frac{1}{2}$, but $d_{C_\infty}(\bar{x}) = 1$. A fortiori, we again have an example of nested sets which fail to converge in the slice topology. □

We recall that $\{C_n\}$ slice converges to C_∞ if and only if $\{C_n\}$ Wijsman converges to C_∞ for every equivalent norm on X ([1], and private communication). In light of Theorem 1 and Remark 5(g) it is reasonable to ask when the linear sets

$$C_n := \{x \in X \ : \ \langle a_k^*, x\rangle = 0 \ \text{ for } \ 1 \leq k \leq n\}$$

converge Wijsman (with respect to a given norm on X) to C_∞. Here, as before, we suppose $\{a_n^*\}$ to be w^*-densely spanning in X^* but not norm-dense.

Let $(X, \| \ \|)$ and $\bar{x} \in X$ be given and observe that

$$\sigma_n := \tfrac{1}{2} d_{C_n}(\bar{x})^2 = \inf_X \left\{\tfrac{1}{2}\|x\|^2 \ : \ \langle a_k^*, x\rangle = \langle a_k^*, \bar{x}\rangle, \ 1 \leq k \leq n\right\}$$

which has Fenchel dual

$$\sigma_n = \max\left\{\sum_{k=1}^{n} \lambda_k \langle a_k^*, \bar{x}\rangle - \tfrac{1}{2} \, \Big\|\sum_{k=1}^{n} \lambda_k a_k^*\Big\|^2\right\},$$

$$= \max\left\{\langle p, \bar{x}\rangle - \tfrac{1}{2} \, \|p\|_*^2 \ : \ p \in \text{span}\{a_1^*, \ldots, a_n^*\}\right\},$$

so that, with Y as before

$$\sup_{n\to\infty} \sigma_n = \lim_{n\to\infty} \sigma_n = \sup\left\{\langle y, \bar{x}\rangle - \tfrac{1}{2} \, \|y\|_*^2 \ : \ y \in Y\right\}.$$

Since

$$\tfrac{1}{2} d_{C_\infty}(\bar{x})^2 = \tfrac{1}{2}\|\bar{x}\|^2 = \sup\left\{\langle y, \bar{x}\rangle - \tfrac{1}{2} \, \|y\|_*^2 \ : \ y \in X^*\right\},$$

it is apparent that Wijsman convergence fails in $\| \ \|$ (at $\bar{x}$) precisely if

$$\sup\left\{\langle y, \bar{x}\rangle - \tfrac{1}{2} \, \|y\|_*^2 \ : \ y \in Y\right\}$$
$$< \tfrac{1}{2} \, \|\bar{x}\|^2 = \sup\left\{\langle y, \bar{x}\rangle - \tfrac{1}{2} \, \|y\|_*^2 \ : \ y \in X^*\right\}$$

or, equivalently, if $Y = \overline{\text{span}}\{a_n^*,\ n = 1, 2, \ldots\}$

$$\sup\{\langle y, \bar{x}\rangle\ :\ \|y\|_* \leq 1,\ y \in Y\} \\ < \|\bar{x}\| = \sup\{\langle y, \bar{x}\rangle\ :\ \|y\|_* \leq 1,\ y \in X^*\}\,. \qquad (**)$$

Thus, it remains to determine when $(**)$ holds for $\{a_n^*\}$ as in $(*)$: that is to say, an example exists precisely unless is Y (1-) *norming* for X. We recast this as:

THEOREM 6. *Let $(X, \|\ \|)$ be a given normed space. As before suppose $\{a_n^*\}$ to be w^*-densely spanning in X^* with*

$$Y := \overline{\text{span}}\{a_n^*,\ n = 1, 2, \ldots\} \neq X^*\,.$$

The linear sets

$$C_n := \{x \in X\ :\ \langle a_k^*, x\rangle = 0\ \text{ for }\ 1 \leq k \leq n\}$$

converge Wijsman (with respect to the given norm on X) to $C_\infty \doteq \{0\}$ if and only if Y is a norming subspace of X^*.

EXAMPLES 7. (a) Consider, the case of $L^1([0,1])$ and the w^*-dense subspace $Y := C([0,1])$, arising from Hausdorff or Fourier moments. We will show Y is norming in $\|\ \|_1$ and so the corresponding sequences which fail to converge in slice topology are always Wijsman convergent.

Fix $\bar{x}$ in L^1 and $\varepsilon > 0$. We select $0 \neq m$ in L^∞ with $\|\bar{x} - m\|_1 < \varepsilon/3$. Now we construct a continuous function g with $\|g - \text{sign}(m)\|_1 < \varepsilon/(3\|m\|_\infty)$ and with $\|g\|_\infty \leq 1$. Then

$$\int_0^1 g(t)\,\bar{x}(t)\,\mathrm{d}t \\ \geq \int_0^1 g(t)\,m(t)\,\mathrm{d}t - \frac{\varepsilon}{3} \geq \int_0^1 \text{sign}(m)(t)\ m(t)\,\mathrm{d}t - \frac{2\varepsilon}{3} \\ = \|m\|_1 - \frac{2\varepsilon}{3} \geq \|\bar{x}\|_1 - \varepsilon.$$

Since $\|g\|_\infty \leq 1$, $C([0,1])$ is norming as claimed.

(b) Suppose Y is a nonreflexive separable normed space and $X := Y^*$ while the functionals $\{a_n^*\}$ have norm-dense span in Y. Then Y is norming and so the corresponding sets $\{C_n\}$ will be Wijsman convergent but will fail to be slice convergent. In particular, such is the case for $X := l^1$ and $Y := c_0$.

(c) Suppose X is a separable space with a nonseparable dual (i.e. X is a separable non-Asplund space). We may always construct a separable norming subspace Y in X^* by fixing a countable dense subset $\{x_n\ :\ n = 1, 2, \ldots\}$ in X

and taking sequences of norm-one linear functionals $\{f_m^n : n, m = 1, 2, \dots\}$ with $\sup_m \langle f_m^n, x_n \rangle = \|x_n\|$. Then $Y := \overline{\mathrm{span}}\{f_m^n : n, m = 1, 2, \dots\}$ is a separable norming subspace in X^*.

(d) More precisely, if we fix $\bar{x} \in X$ with $\|\bar{x}\| = 1$ and select $F \in X^{**} \backslash X$ so that $\|F - \bar{x}\| < \alpha < 1$, as we always may arrange, then $Y := N(F) = \{x^* \in X^* : \langle F, x^* \rangle = 0\}$ satisfies

$$\sup \{\langle y, \bar{x} \rangle : \|y\| \leq 1, \ y \in Y\} \leq \alpha < \|\bar{x}\| = \sup \{\langle y, \bar{x} \rangle : \|y\| \leq 1, \ y \in X^*\} .$$

If, as in Theorem 1, $\{a_n^*\}$ is chosen w^*-dense in $N(F)$, then we see that in any nonreflexive space (separable or not) we may construct nested linear sets $\{C_n\}$ such that $\{C_n\}$ is not Wijsman convergent in direction $\bar{x}$.

(e) Consider $X := c_0$ and take $\{a_n^*\}$ to be $\{e_n^*\}$, the coordinate functionals. Then the corresponding $\{C_n\}$ do slice-converge. This illustrates that in Theorem 1 and Theorem 6, it is essential that Y not equal X^*. Moreover, we observe that, in the uniform norm, c_0 admits no nontrivial (1-)norming subspaces. □

EXAMPLE 8. (a) Suppose that X is a separable nonreflexive Banach lattice with positive cone X_+. Many of the most interesting concrete sequences arising in estimation theory are realizable as

$$C_n^+ := \{x \in X_+ : \langle a_k^*, x - \bar{x} \rangle = 0 \ \text{ for } \ 1 \leq k \leq n\}$$

where $\bar{x} \in X_+$, and as before $\{a_n^*\}$ is w^*-densely spanning in X^* with $Y := \overline{\mathrm{span}}(a_n^*, \ n = 1, 2, \dots\} \neq X^*$. Assume in addition that Y is a sub-lattice.

Much as in the argument of Theorem 6

$$\sigma_n := \tfrac{1}{2} d_{C_n^+}(0)^2 = \inf_X \{\tfrac{1}{2}\|x\|^2 : \langle a_k^*, x \rangle = \langle a_k^*, \bar{x} \rangle, \ 1 \leq k \leq n, \ x \in X_+\}$$

which has Fenchel dual

$$\sigma_n = \max \left\{ \sum_{k=1}^{n} \lambda_k \langle a_k^*, \bar{x} \rangle - \tfrac{1}{2} \left\| \left(\sum_{k=1}^{n} \lambda_k a_k^* \right)^+ \right\|^2 : \lambda_k \geq 0, \ 1 \leq k \leq n \right\}$$

$$= \max \{\langle p, \bar{x} \rangle - \tfrac{1}{2} \|p^+\|_*^2 : p \in \mathrm{span}\{a_1^*, \dots, a_n^*\} \} .$$

This requires an easily satisfied constraint qualification:

$$\exists \hat{x} \in \mathrm{qi}\, X_+ \ \text{ with } \ \langle a_k^*, \hat{x} \rangle = \langle a_k^*, \bar{x} \rangle \ \text{ for } \ 1 \leq k \leq n . \qquad (\mathrm{CQ}_n^+)$$

Here 'qi' denotes the *quasi-interior*. In specific cases, this is easy to validate.

Thus

$$\sigma := \sup_{n \to \infty} \sigma_n = \lim_{n \to \infty} \sigma_n = \sup \{\langle y, \bar{x} \rangle - \tfrac{1}{2} \|y^+\|_*^2 : y \in Y\} ,$$

since the lattice operations are norm-continuous. However as $\bar{x} \in X_+$

$$\begin{aligned}\sigma &= \sup\{\langle y,\bar{x}\rangle - \tfrac{1}{2}\|y^+\|_*^2 \ : \ y \in Y\} \\ &\leq \sup\{\langle y^+,\bar{x}\rangle - \tfrac{1}{2}\|y^+\|_*^2 \ : \ y \in Y\} \\ &\leq \sup\{\langle y,\bar{x}\rangle - \tfrac{1}{2}\|y\|_*^2 \ : \ y \in Y\} \ \ (\text{as } Y^+ \subset Y) \\ &\leq \sup\{\langle y,\bar{x}\rangle - \tfrac{1}{2}\|y^+\|_*^2 \ : \ y \in Y\} = \sigma\,,\end{aligned}$$

where the last inequality follows since $\| \ \|$ is a lattice norm. Hence, as before, $\sigma = \sup\{\langle y,\bar{x}\rangle - \frac{1}{2}\|y\|_*^2 \ : \ y \in Y\}$ and it is again apparent that Wijsman convergence of $\{C_n^+\}$ fails in $\| \ \|$ (at 0) precisely when Y is not 1-norming in direction $\bar{x}$.

(b) As in Example 7(a), consider the case of $L^1([0,1])$. Let $\bar{x} \geq 0$ and $Y := N(F)$ be as in Example 7(d) and consider the corresponding $\{C_n^+\}$.

Whenever Y is a sub-lattice of X^*, the argument of part (a) applies to show that Wijsman convergence fails at zero: $\lim_{n\to\infty} d_{C_n^+}(0) \neq \|\bar{x}\|$. This requires checking that (CQ_n^+) holds for all n as is the case when $\bar{x} > 0$ a.e.; but in the case of Hausdorff moments, or other real analytic moments, it suffices that the system of equations have a nonzero, nonnegative solution ([5]).

(c) Equivalently, such nonnorming functions $\bar{x}$ give rise to sequences of moment-matching L^1-optimization problems

$$p_n := \min\left\{\|x\|_1 : \int_0^1 [x(t) - \bar{x}(t)]\, a_k(t)\,dt = 0 \ \text{ for } \ 1 \leq k \leq n, \ x \geq 0 \text{ a.e.}\right\}$$

so that the optimal values $\{p_n\}$ do not converge back to the correct limit value $\|\bar{x}\|$. Notice, by contrast, that for Hausdorff moments ($a_k(t) := t^{k-1}$) we must have $p_n \equiv \|\bar{x}\|$. Of course, now Y is 1-norming in all directions. □

The attentive reader will notice that we have not actually constructed a densely spanning non(1-)norming sublattice in either (a) or (b) of the previous example. We finish by showing how to do so with a construction suggested by Mr Xiaopeng Gao of the University of Western Ontario.

EXAMPLE 9. (a) (An example of a sublattice of $C[0,1]$ that is total but not norming for $L^1[0,1]$.)

Let $F \subset [0,\frac{1}{3}]$ and $G \subset [\frac{2}{3},1]$ be (closed) Cantor sets with $\mu(F) > 0$ and $\mu(G) = 0$. Here μ is Lebesgue measure. Let $\psi \ : \ [0,1] \to [0,1]$ be continuous, one-to-one and onto, and have $\psi(F) = G$. Fix $\gamma > 0$ and define

$$Y^\gamma := \{f \in C[0,1] \ : \ f(t) = \gamma f(\psi(t)), \ \forall t \in F\}\,.$$

Then

(i) Y^γ is a closed subspace of $C[0,1]$ and $|y| \in Y^\gamma$ whenever $y \in Y^\gamma$. Thus, Y^γ is a Banach sublattice of $C[0,1]$, and so (a.e.) of $L^\infty[0,1]$.

(ii) Define

$$\|x\|_\gamma := \sup \langle f, x \rangle \ : \ \|f\|_\infty \leq 1 \ \text{ and } \ f \in Y^\gamma\} \, .$$

Then

$$\|\chi_F\|_\gamma = \sup \left\{ \int_F f \ : \ \|f\|_\infty \leq 1 \ \text{ and } \ f \in Y^\gamma \right\} \leq \gamma\mu(F) = \gamma\|\chi_F\|_1$$

since $f(\psi(x)) \leq \gamma$ for x in F. Hence Y^γ is not better than γ-norming.

(iii) Let $f \in C[0,1]$ and $\frac{1}{3} > \varepsilon > 0$ be given. Let f_ε be defined by

$$f_\varepsilon(t) := \begin{cases} f(t) \, , & \text{if } \ d_G(t) \geq \varepsilon \, , \\ \gamma^{-1} f(\psi^{-1}(t)) \, , & \text{if } \ t \in G \, , \end{cases}$$

where, by the Tietze-Katetov extension theorem, f_ε is extended continuously with $\|f_\varepsilon\|_\infty = \gamma^{-1}\|f\|_\infty$. Then f_ε is in Y^γ by construction.

Suppose that $\hat{x}$ in L^1 is annihilated by Y^γ. Then $\int f_\varepsilon \hat{x} \, \mathrm{d}\mu = 0$ for all $\varepsilon > 0$. Hence

$$|\int f\hat{x} \, \mathrm{d}\mu| = |\int (f - f_\varepsilon) \, \hat{x} \, \mathrm{d}\mu| \leq 2\gamma^{-1} \int_{d_G(t)<\varepsilon} |\hat{x}| \, \mathrm{d}\mu \to 0$$

since $\mu\{t \ : \ d_G(t) < \varepsilon\} \to 0$, with ε, because G is closed and null, and as $\hat{x}$ is in L^1. Thus $\hat{x}$ is annihilated by $C[0,1]$ and so is zero. Hence, Y^γ is a total sublattice of L^∞ but as desired is not 1-norming.

(b) Somewhat more elaborately, we can (essentially) set $Y := \bigcap_{\gamma>0} Y^\gamma$ on choosing appropriate disjoint Cantor sets, and arrange that the sublattice is total but not γ-norming for any $\gamma > 0$. □

References

1. Aubin, J.-P. and Frankowska, H.: *Set Valued Analysis*, Birkhaüser, Basle, 1990.
2. Beer, G.: The slice topology, a viable alternative to Mosco convergence in nonreflexive spaces, *J. Nonlinear Anal.: Theory Methods, Appl.* **19** (1992), 271–290.
3. Beer, G. and Borwein, J.: Mosco convergence of level sets and graphs of linear functionals, *J. Math. Anal. Appl.*, accepted November 1991.
4. Beer, G. and Borwein, J.M.: Mosco convergence and reflexivity, *Proc. Amer. Math. Soc.* **109** (1990), 427–436.
5. Borwein, J.M. and Lewis, A.S.: Partially-finite convex programming in L^1: entropy maximization, *SIAM J. Optim.* **3** (1993), in press.
6. Borwein, J.M. and Lewis, A.S.: Partially-finite convex programming, (I), (II), *Math. Programming, Series B* **57** (1992), 15–48, 49–84.
7. Borwein, J.M. and Lewis, A.S.: Strong rotundity and optimization, *SIAM J. Optim.*, accepted July 1992. [Research Report CORR 91–16].
8. Rockafelar, R.T.: *Conjugate Duality and Optimization*, SIAM, Philadelphia, 1974.