Systems Engineering 520

Linear Programming

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Lecture 15

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Overview

- Definition of a linear program
- Geometric interpretation of a linear program and the general idea of the simplex algorithm
- Dual of a linear program
- Transportation problem
- Assignment problem
- Network flow problem
- Shortest path problem
- Max flow problem

Linear Program

- A linear program is an optimization problem with linear objective function and linear constraints

$$
\begin{array}{lll}
\max & \displaystyle\sum_{j=1}^{n} c_j \, x_j & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} \, x_j \leq b_i & i = 1, \ldots, m \\
& x_j \geq 0 & j = 1, \ldots, n
\end{array}
$$

Linear Programs

- A common way to write a linear program is to use the matrix/vector notation

$$\begin{aligned} \max \quad & c\,x \\ \text{subject to} \quad & A\,x \leq b \\ & x \geq 0 \end{aligned}$$

- x, b and c are vectors and A is a matrix
- The constraints can be of “equality” or “greater than or equal to” type

Why Do We Care about Linear Programs?

- We only allow linear objective functions and linear constraints in linear programs
- This seems very restrictive
- We care about linear programs because
  - A nonlinear programming problem with a convex objective function can be approximated with a linear program
  - Linear programs are very well-understood
  - We have very robust solution algorithms for linear programs
  - Life is often linear

A Toy Example

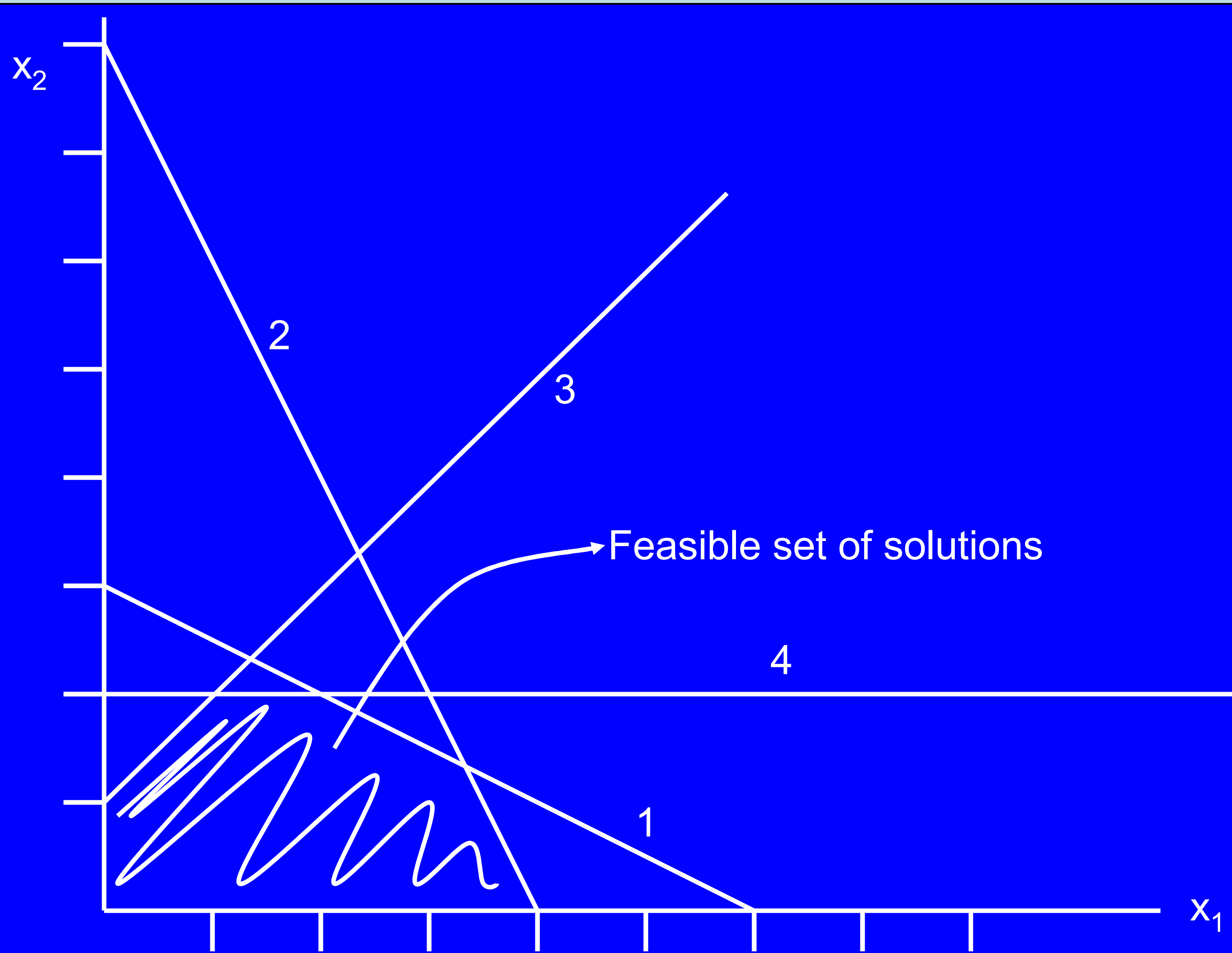
- A paint company produces two types of paints (exterior and interior) by using two types of raw materials (material A and material B)
- Each ton of exterior paint uses 1 ton of material A and 2 tons of material B
- Each ton of interior paint uses 2 tons of material A and 1 ton of material B
- 6 tons of material A and 8 tons of material B are available
- The profit per ton of exterior paint is 3 and the profit per ton of interior paint is 2
- The company is interested how much of each type of paint to produce

A Toy Example

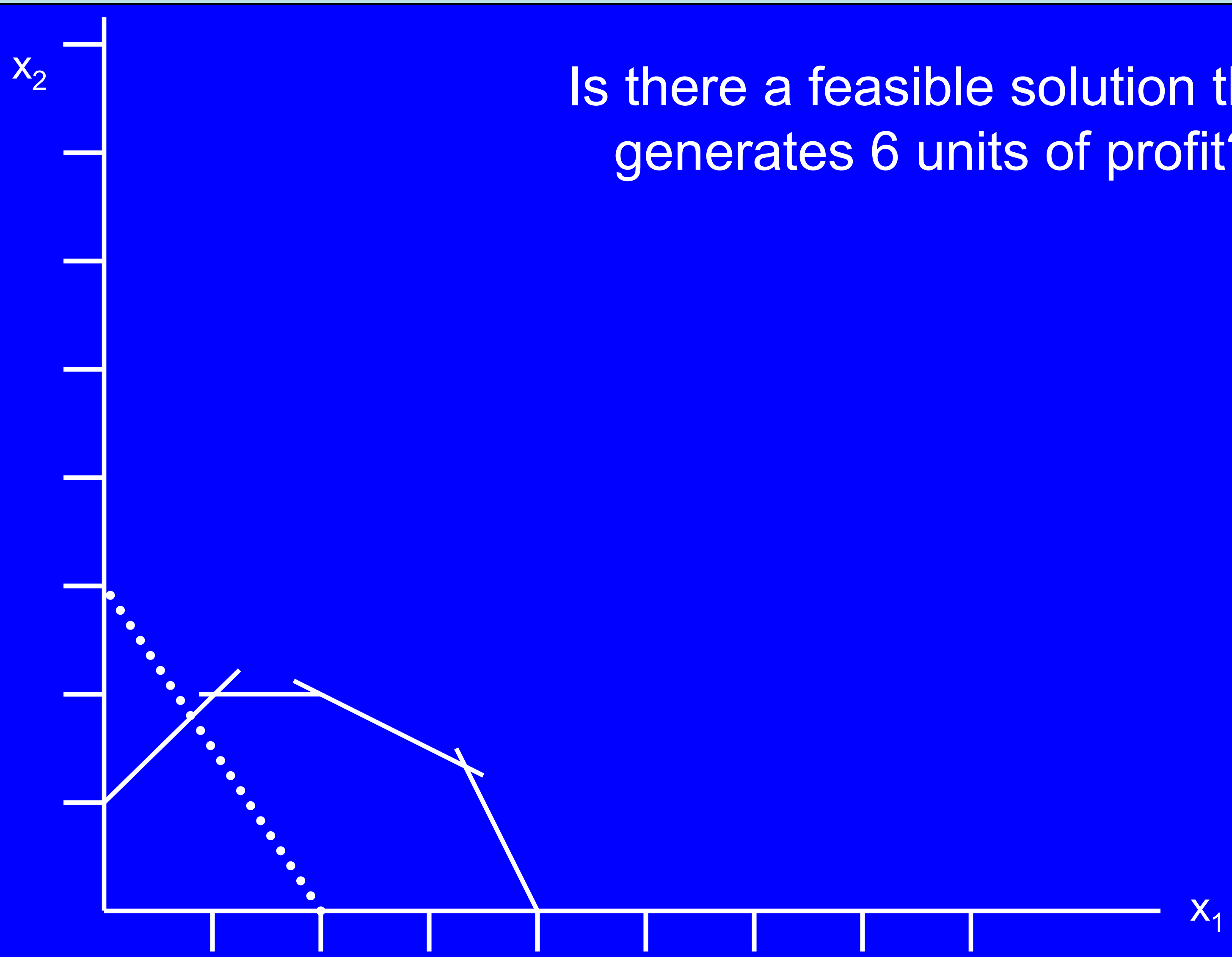
- The company can sell at most 2 tons of interior paint
- The amount of interior paint produced cannot exceed the amount of exterior paint by more than 1 ton

$$\begin{aligned} \max \quad & 3\,x_1 + 2\,x_2 \\ \text{subject to} \quad & x_1 + 2\,x_2 \leq 6 \\ & 2\,x_1 + x_2 \leq 8 \\ & x_2 - x_1 \leq 1 \\ & x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Geometric Interpretation

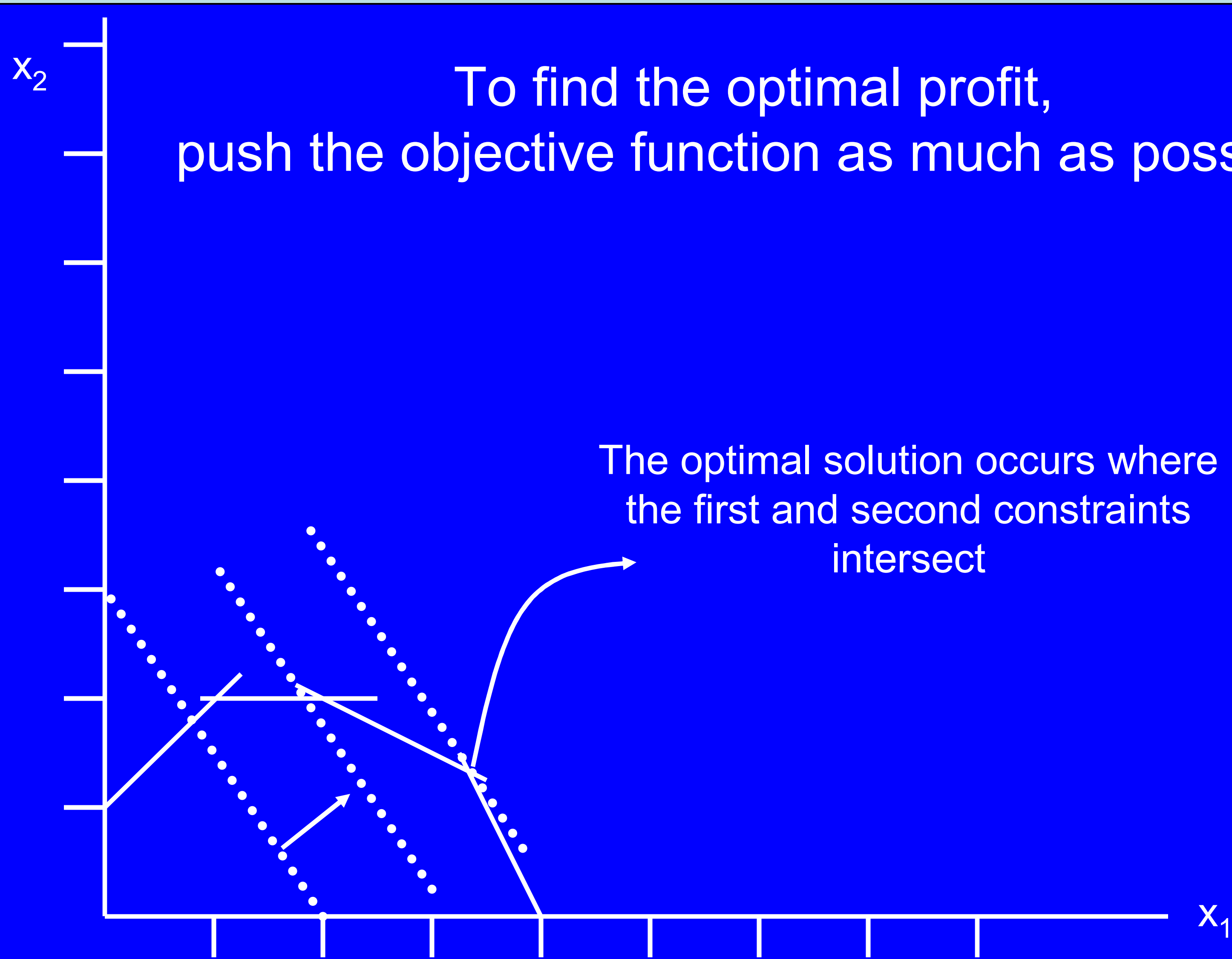


Geometric Interpretation



Is there a feasible solution that generates 6 units of profit?

Geometric Interpretation



A Toy Example

- To obtain the optimal solution, solve

  $x_1 + 2\,x_2 = 6$ and $2\,x_1 + x_2 = 8$
- We obtain $x_1^* = 10/3$ and $x_2^* = 4/3$
- It is not hard to see that the optimal solution always occurs at an extreme point (a corner)
- How do we find the extreme points?

A Toy Example

- How do we find the extreme points?
- Roughly speaking, if the problem has m constraints and n decision variables, then take n of the constraints and solve for the values of n decision variables
- This gives an extreme point
- To check all extreme points, we need to try (m choose n) possible cases, which can be a huge number!
- Simplex algorithm is an efficient way of moving from one extreme point to another one so that the value of the objective function increases at each move
- Gradient search and penalty functions are not needed when solving linear programs

Dual of a Linear Program

- Try to apply the duality ideas to a linear program

$$\begin{array}{lll} \max & \displaystyle\sum_{j=1}^{n} c_j \, x_j & \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} \, x_j \leq b_i & i = 1, \ldots, m \\ & x_j \geq 0 & j = 1, \ldots, n \end{array}$$

- Associate the Lagrange multipliers λ_i $i = 1,\ldots,m$ with the constraints to construct the Lagrange function
- Do not relax the nonnegativity constraints

Dual of a Linear Program

$$\begin{aligned} L(x,\lambda) &= \sum_{j=1}^{n} c_j\, x_j + \sum_{i=1}^{m} \lambda_i \left[b_i - \sum_{j=1}^{n} a_{ij}\, x_j \right] \\ &= \sum_{j=1}^{n} \left[c_j - \sum_{i=1}^{m} a_{ij}\, \lambda_i \right] x_j + \sum_{i=1}^{m} b_i\, \lambda_i \end{aligned}$$

- The dual function is

$$\begin{aligned} D(\lambda) &= \max_{x \geq 0} L(x,\lambda) \\ &= \max_{x \geq 0} \sum_{j=1}^{n} \left[c_j - \sum_{i=1}^{m} a_{ij}\, \lambda_i \right] x_j + \sum_{i=1}^{m} b_i\, \lambda_i \end{aligned}$$

Dual of a Linear Program

- Similar to our earlier results, the dual function is an upper bound on the optimal objective value of the problem
- (Since we are maximizing, the dual function gives an upper bound, not a lower bound)
- To obtain the tightest upper bound, we solve the problem

$$\min_{\lambda \geq 0} D(\lambda)$$

Dual of a Linear Program

- Note that the dual function is infinite when

$$c_j - \sum_{i=1}^{m} a_{ij}\,\lambda_i > 0$$

for some j, whereas the dual function is

$$D(\lambda) = \sum_{i=1}^{m} b_i\,\lambda_i$$

when

$$c_j - \sum_{i=1}^{m} a_{ij}\,\lambda_i \leq 0$$

Dual of a Linear Program

- Therefore, solving the problem

$$\min_{\lambda \geq 0} D(\lambda)$$

is equivalent to solving the linear program

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} b_i \, \lambda_i & \\ \text{subject to} \quad & \sum_{i=1}^{m} a_{ij} \, \lambda_i \geq c_j & j = 1, \ldots, n \\ & \lambda_i \geq 0 & i = 1, \ldots, m \end{aligned}$$

Dual of a Linear Program

- It is possible to prove that the optimal objective value of the dual linear program is equal to the optimal objective value of the original linear program
- The number of constraints in the dual linear program is equal to the number of decision variables in the original linear program
- The number of decision variables in the dual linear program is equal to the number of constraints in the original linear program
- Therefore, instead of solving a linear program with a large number of decision variables and a small number of constraints, we can solve a linear program with a small number of decision variables and a large number of constraints

Transportation Problem

- We have 3 warehouses and 2 retail centers
- It costs c_{ij} to ship a unit of product from warehouse i to retail center j
- Inventory at warehouse i is s_i
- Demand at retail center j is d_j
- For simplicity, first assume that $s_1 + s_2 + s_3 = d_1 + d_2$ so that total supply equals total demand
- What is the cheapest way of shipping the products from warehouses to the retailers?

Transportation Problem

- What do we do when we do not have $s_1 + s_2 + s_3 = d_1 + d_2$?

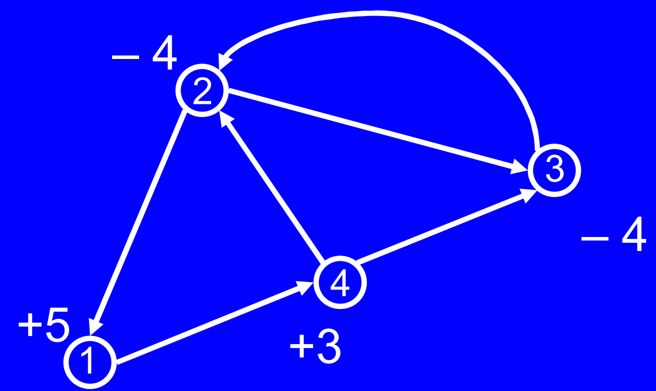
- If all supply and demand data are integers, then there exists an integer-valued optimal solution to the transportation problem

Assignment Problem

- We have 3 technicians and 3 jobs
- Suitability of technician i for job j is c_{ij}
- Each job requires one technician and each technician can work at most one job
- What is the best way of assigning technicians to jobs to maximize the total suitability?
- What do we do when the number of technicians is not equal to the number of jobs?
- There exists an integer-valued optimal solution to the assignment problem even if we do not explicitly impose the integrality constraints

Network Flow Problem

- We have a network with 4 nodes and 6 arcs



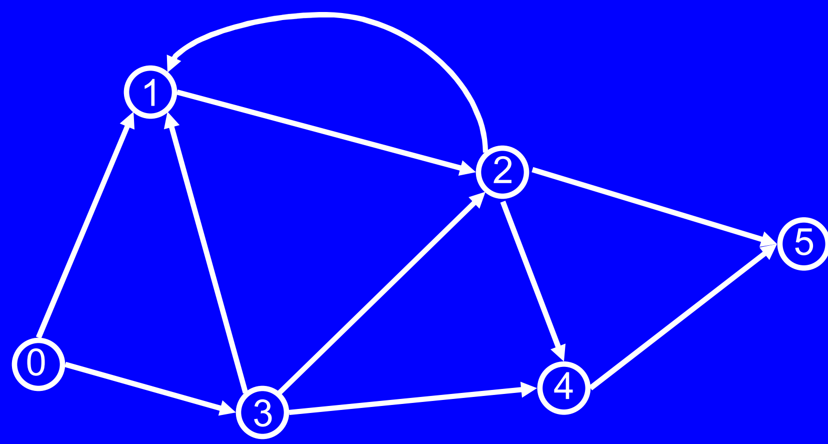
- The signed numbers next to the nodes are the supplies and demands at each node
- Positive numbers are supplies and negative numbers are demands
- Cost of shipping a unit of supply over arc (i,j) is c_{ij}
- What is the cheapest way of shipping the supplies from the supply nodes to the demand nodes?

Network Flow Problem

- What do we do when the total supply is not equal to total demand?
- There can be upper bounds on how much we can ship on the arcs
- If all supply, demand and upper bound data are integers, then there exists an integer-valued optimal solution to the network flow problem

Shortest Path Problem

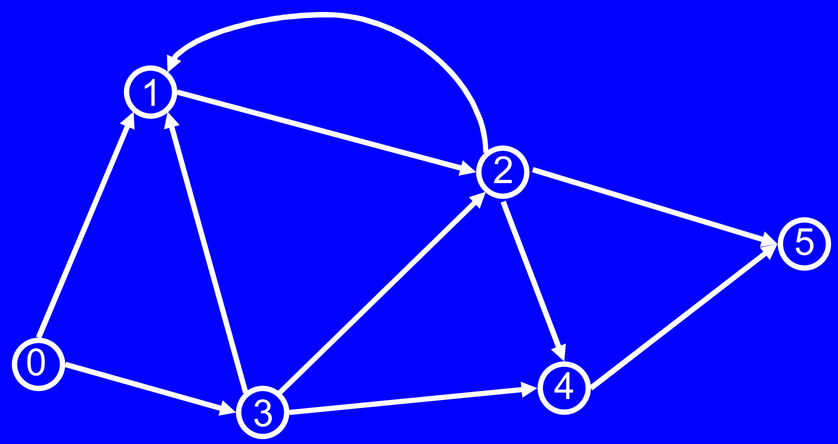
- We have a network with 6 nodes and 10 arcs



- Cost of moving over arc (i,j) is c_{ij}
- What is the cheapest way to go from node 0 to node 5?
- Rephrase the problem: Node 0 has 1 unit of supply and node 5 has one unit of demand
- What is the cheapest way of shipping the supply from the supply node to the demand node?

Maximum Flow Problem

- We have a network with 6 nodes and 10 arcs



- The upper bound over arc (i,j) is u_{ij}
- Assume that there is infinite supply available at node 0 and there is infinite demand at node 5
- What is the maximum amount of supply that we can ship from node 0 to node 5?