Systems Engineering 520

Output Analysis of Discrete-Event Simulation Models

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Lecture 7

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Overview

- Terminating and steady state simulation models
- Output analysis of terminating simulation models
- Output analysis of steady state simulation models
- Comparison of two simulation models

Terminating and Steady State Simulation

- A terminating simulation model runs for a certain duration of time, and then, stops
- The duration of time for which the model runs may be random
- E.g. consider a model that simulates the operation of a bank from 8:00 am to 4:30 pm (the duration is deterministic)

  Consider a model that simulates the operation of a restaurant until the last party leaves (the duration is random)

Terminating and Steady State Simulation

- A steady state simulation runs continuously without any well-defined stopping condition
- We are interested in the long run (or steady state) behavior of the system
- E.g. consider a model that simulates the operation of a production facility that runs continuously

  Assume that the conditions of the production processes (e.g. demand rates, machine capacities, etc.) remain stable

  We are interested in the long term average rate of production

Terminating and Steady State Simulation

- Consider a model that simulates the behavior of a communication network

  We assume that the demand remains stable over a time interval of 10-20 minutes

  Data packets arrive very frequently (e.g. millions per second)

  We can view a simulation over a few minutes as being essentially over an infinite time horizon

  We are interested in the average long run congestion in the network

Output Analysis of Terminating Simulation Models

- Standard confidence interval methodology applies when analyzing the output of a terminating simulation model
- Assume that we run the model for R replications
- Each replication uses an independent sequence of random numbers and starts from the same initial conditions
- Let the trajectory of the model in replication r be

$$\{Y_r(t) : t \in [0, T]\}$$

- We can think of $Y_r(t)$ as the length of a particular queue at time t of replication r

Output Analysis of Terminating Simulation Models

- Assume that we are interested in the performance measure

$$\theta = \mathbb{E}\left\{ \frac{1}{T} \int_0^T Y_r(t)\, dt \right\}$$

- If we think of $Y_r(t)$ as the length of a particular queue at time t of replication r, then the performance measure we are interested in is the expected average length of the queue over T time periods

Output Analysis of Terminating Simulation Models

- From replication r of the simulation model, we obtain an estimate of the performance measure

$$\bar{Y}_r = \frac{1}{T} \int_0^T Y_r(t)\, dt$$

- To build a confidence interval for the performance measure, we compute

$$\bar{Y} = \frac{1}{R} \sum_{r=1}^{R} \bar{Y}_r$$

$$s_R^2 = \frac{1}{R-1} \sum_{r=1}^{R} (\bar{Y}_r - \bar{Y})^2$$

Output Analysis of Terminating Simulation Models

- In this case, 100(1 – α)% confidence interval is given by

$$\bar{Y} \mp z_{\alpha/2} \frac{s_R}{R^{1/2}}$$

- Assume that we run the single-server queueing model for 40 runs

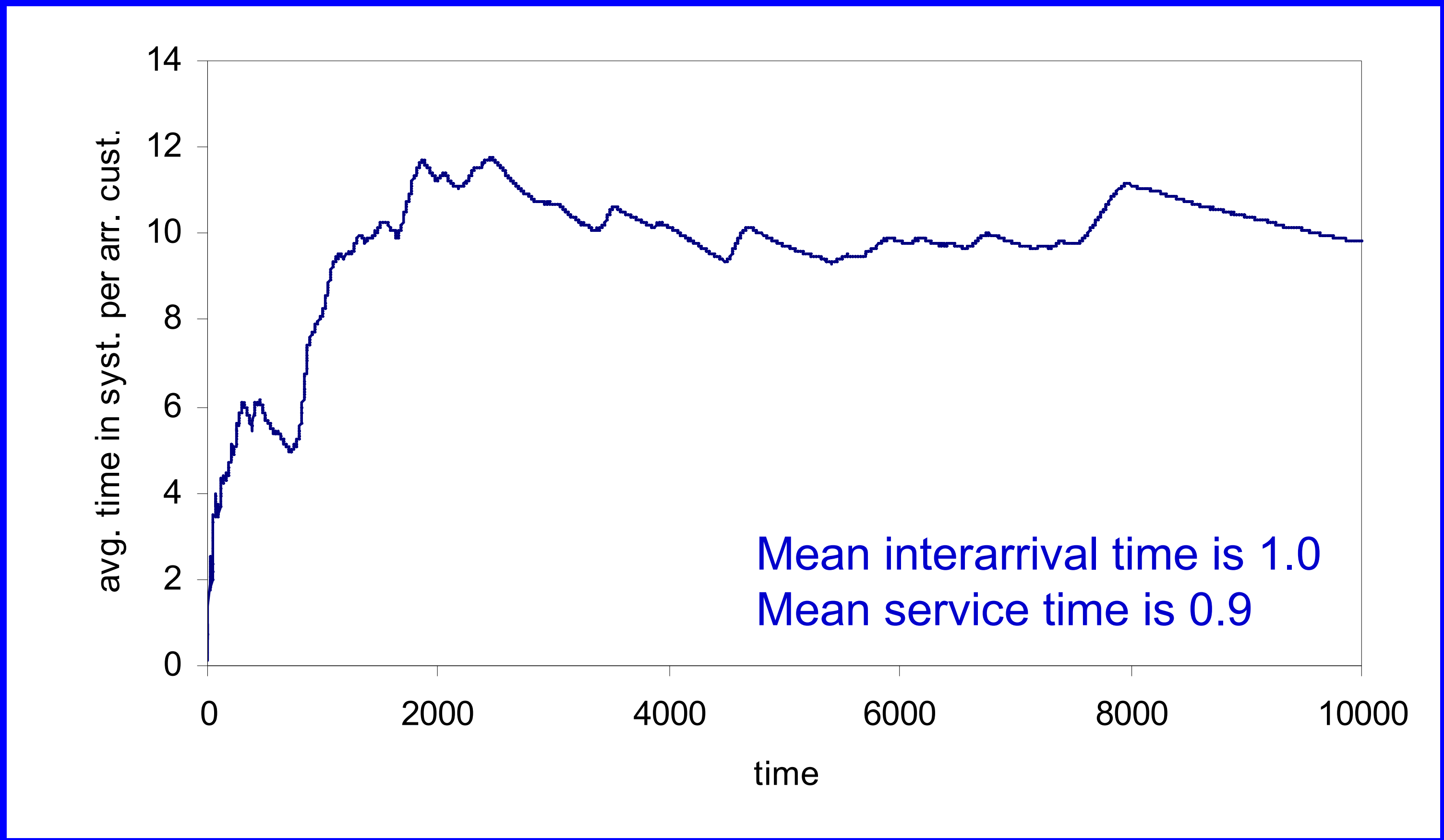
	A	B	C	D	E
1	Run no	No arrivals	No departures	Average time in system	
2	1	30	30	0.338315743	
3	2	43	41	0.68862269	
4	3	36	35	0.659076869	
5	4	51	51	0.770093249	
6	5	43	43	0.839267263	

Output Analysis of Steady State Simulation Models

- We may be interested in steady state performance for several reasons
- When working with a terminating simulation model, the performance measures depend on how the initial conditions and the termination time are specified
- When working with a steady state simulation model, the long run performance measures do not depend on how the initial conditions are specified
- Long run performance measures are easier to compute analytically

Initial Transient Problem

- Plot the average time spent in the system per arriving customer as a function of time



Initial Transient Problem

- To deal with the initial transient problem, run the model for a certain duration of time without collecting statistics
- Run R replications of the simulation model by warming up the model for d time units
- The estimate of the performance measure from run r is

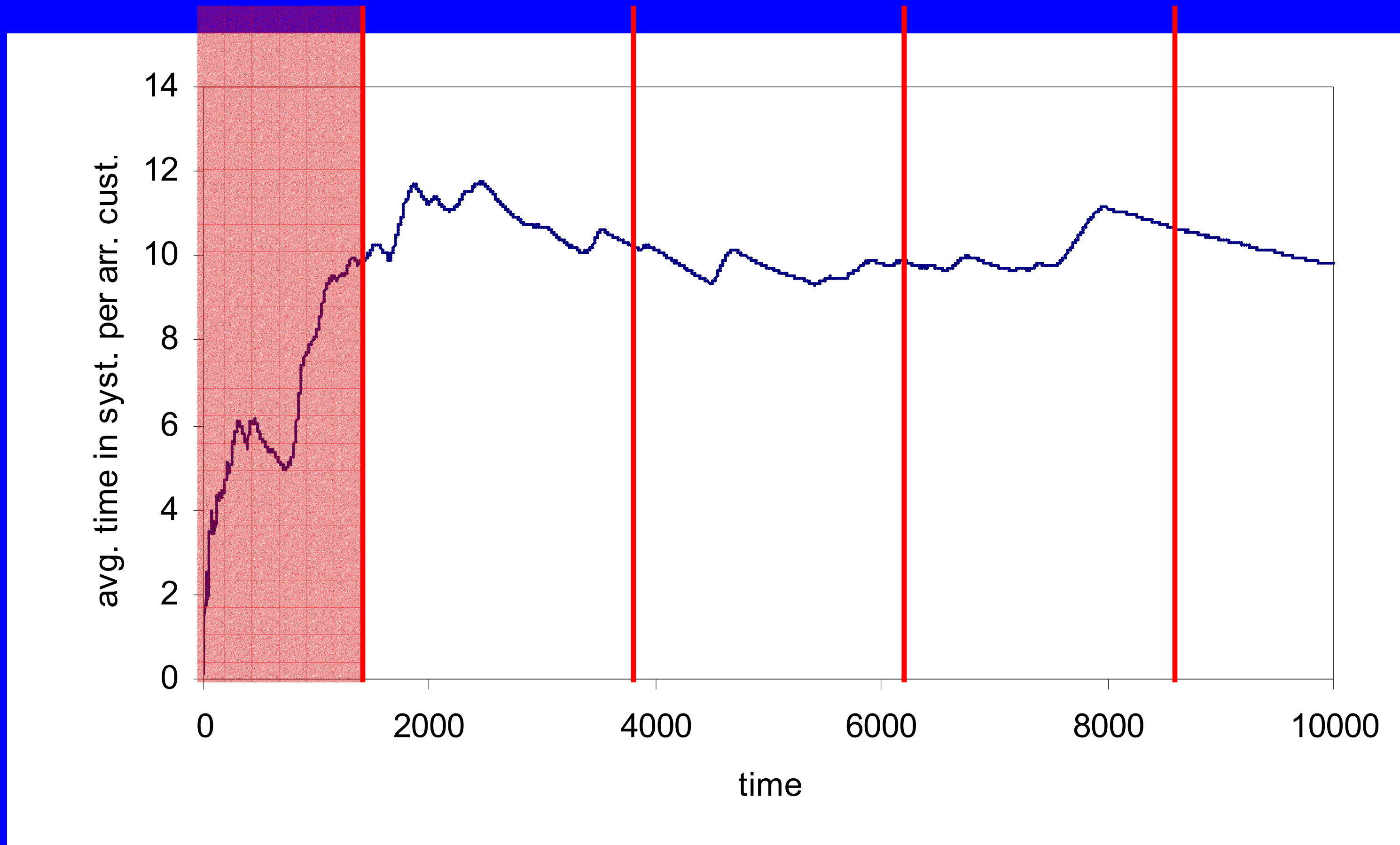
$$\bar{Y}_r = \frac{1}{T-d} \int_d^T Y_r(t)\, dt$$

- Apply the same confidence interval methodology

Initial Transient Problem

- This approach wastes computational effort because the warm up period is repeated at the beginning of each replication
- Instead, run one long replication by warming up the model for d time units
- Divide up the one long replication into R intervals and treat each of the intervals as a replication

Initial Transient Problem



Output Analysis of Steady State Simulation Models

- Let the trajectory of the model be

$$\{Y(t) : t \in [0, T + d]\}$$

- Compute $$\bar{Y}_r = \frac{1}{T/R} \int_{d+(r-1)T/R}^{d+r\,T/R} Y(t)\,dt$$

$$\bar{Y} = \frac{1}{R} \sum_{r=1}^{R} \bar{Y}_r \qquad s_R^2 = \frac{1}{R-1} \sum_{r=1}^{R} (\bar{Y}_r - \bar{Y})^2$$

- Apply the same confidence interval methodology

Output Analysis of Steady State Simulation Models

- How to choose the length of the warm up period?
- See the graphical tools and moving average method in Law & Kelton

Comparison of Two Simulation Models

- Consider two single-server queueing systems
- In the first system, the interarrival times are exponentially distributed with mean 0.5 minutes and the service times are exponentially distributed with mean 0.3 minutes
- In the second system, the interarrival times are exponentially distributed with mean 0.55 minutes and the service times are exponentially distributed with mean 0.35 minutes
- We are interested in the average time spent in the system per arriving customer over the first 20 minutes
- Which system is better?

Comparison of Two Simulation Models

- Run both systems for R replications
- Let $\bar{Y}_r$ be the output of replication r for the first system
- Let $\bar{Z}_r$ be the output of replication r for the second system
- Compute the difference in the performances for r replications $D_r = \bar{Y}_r - \bar{Z}_r$
- Apply the same confidence interval methodology by treating $\{D_r : r = 1, \ldots, R\}$ as the data

Comparison of Two Simulation Models

- We run each system for 20 replications

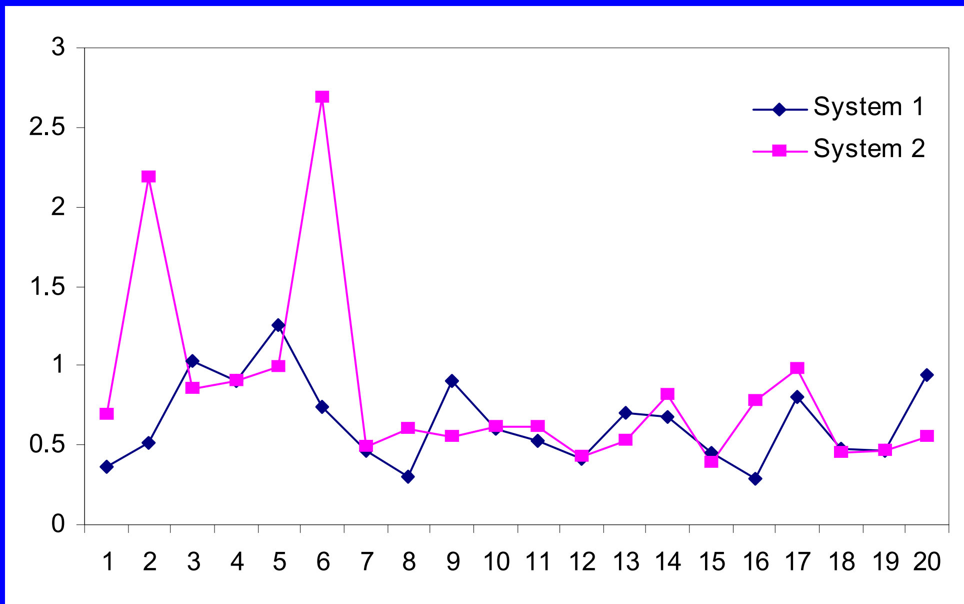
- 95% confidence interval for the difference in the performances of two systems is [-0.44, 0.07]

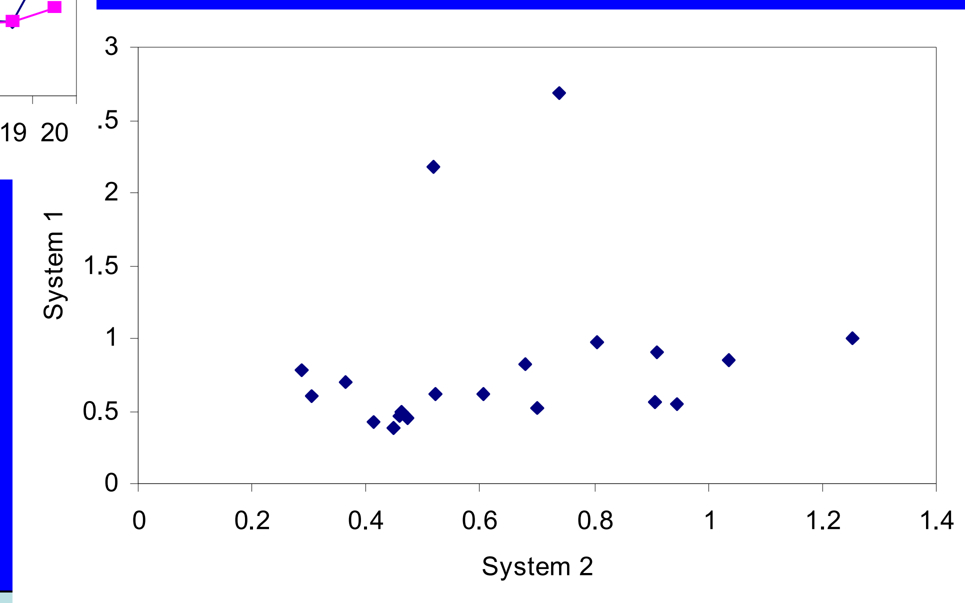
	A	B	C	D
1	System 1	System 2	Difference	Mean
2	0.363423	0.692273	-0.32885	-0.1866
3	0.518737	2.178243	-1.65951	Std. dev
4	1.034803	0.849123	0.18568	0.596068
5	0.90969	0.907258	0.002432	
6	1.252684	0.99397	0.258714	
7	0.738222	2.683232	-1.94501	
8	0.461408	0.488159	-0.02675	
9	0.304291	0.60786	-0.30357	
10	0.904791	0.558119	0.346672	
11	0.604965	0.614002	-0.00904	
12	0.52243	0.613914	-0.09148	
13	0.411624	0.425744	-0.01412	
14	0.69901	0.524365	0.174645	
15	0.679079	0.821119	-0.14204	
16	0.448416	0.384865	0.063551	
17	0.287927	0.774105	-0.48618	
18	0.804498	0.975328	-0.17083	
19	0.471667	0.453733	0.017934	
20	0.45914	0.461216	-0.00208	
21	0.945095	0.547367	0.397728	

Comparison of Two Simulation Models

- Therefore, we can state with 95% confidence that the difference between the performances of two systems lies somewhere between [-0.44, 0.07]
- This statement is inconclusive from the perspective of determining which system is better
- Plot the results of 20 replications for the two systems

Comparison of Two Simulation Models





Comparison of Two Simulation Models

- The replications for the two systems are completely independent of each other
- Imagine for one replication, the first system gets lucky and receives very few arrivals, whereas for the same replication the second system gets unlucky and receives a lot of arrivals
- When the first system’s performance is really good, the second system’s performance can be really bad
- In this case, the difference in the performances of the two systems can be large
- As a result, the standard deviation of the difference becomes large and we obtain wide confidence intervals

Comparison of Two Simulation Models

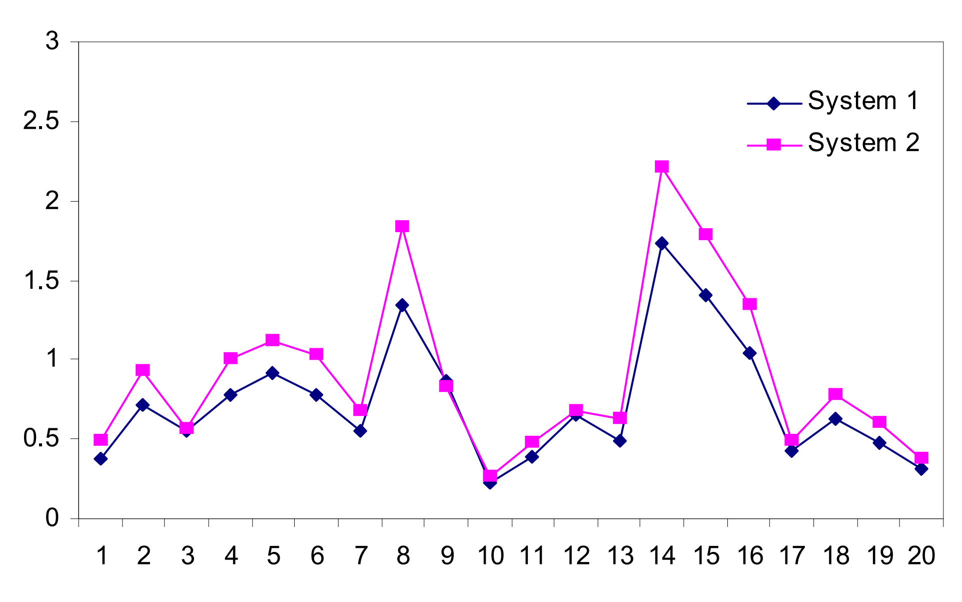
- We can remove the “luck factor” by subjecting the two systems to the same stream of random numbers
- The two systems work with the same stream of random numbers for generating the interarrival times
- The two systems work with the same stream of random numbers for generating the service times

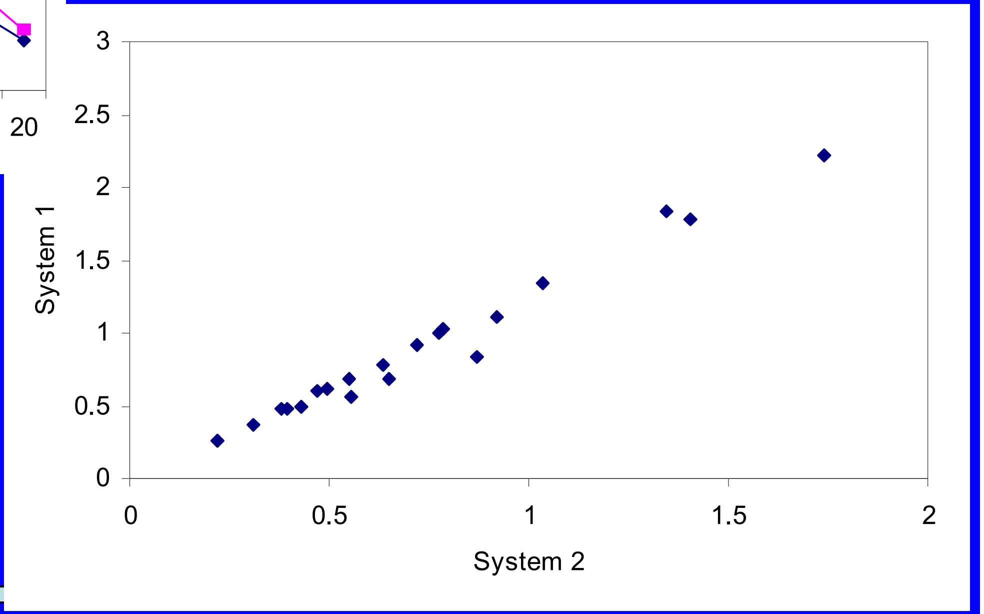
Comparison of Two Simulation Models

- We run each system for 20 replications by using common random numbers
- 95% confidence interval for the difference in the performances of two systems is [-0.23, -0.10]
- We can state with 95% confidence that the first system is better!

F	G	H	I
System 1	System 2	Difference	Mean
0.380808	0.486074	-0.10527	-0.17086
0.721415	0.923136	-0.20172	Std. dev
0.554756	0.566897	-0.01214	0.146923
0.774329	1.00141	-0.22708	
0.919975	1.115714	-0.19574	
0.783589	1.032445	-0.24886	
0.54828	0.683704	-0.13542	
1.343069	1.831995	-0.48893	
0.871269	0.833466	0.037802	
0.219752	0.260614	-0.04086	
0.393203	0.476031	-0.08283	
0.652378	0.683072	-0.03069	
0.493963	0.622886	-0.12892	
1.738406	2.214465	-0.47606	
1.406666	1.78513	-0.37846	
1.03636	1.337378	-0.30102	
0.428433	0.494442	-0.06601	
0.633362	0.774698	-0.14134	
0.470945	0.597696	-0.12675	
0.308855	0.375718	-0.06686	

Comparison of Two Simulation Models





Comparison of Two Simulation Models

- What to do when the two systems are not run for the same number of replications?
- See the statistical tests in Law & Kelton