Systems Engineering 520

Introduction to Optimization and MS Excel's Solver

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Lecture 11

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Optimization

- When we end up with a design, it usually includes certain adjustable parameters
  - Problem is usually deterministic
  - Relationships between the elements of the problem is complex
- When we are running a system, we continuously need to manage the system
  - It is likely that the problem is stochastic
  - Time dimension is important

Simple Example

- Design a box with square base and no top that has the maximum volume
- Surface area should be $500cm^2$
- Variables

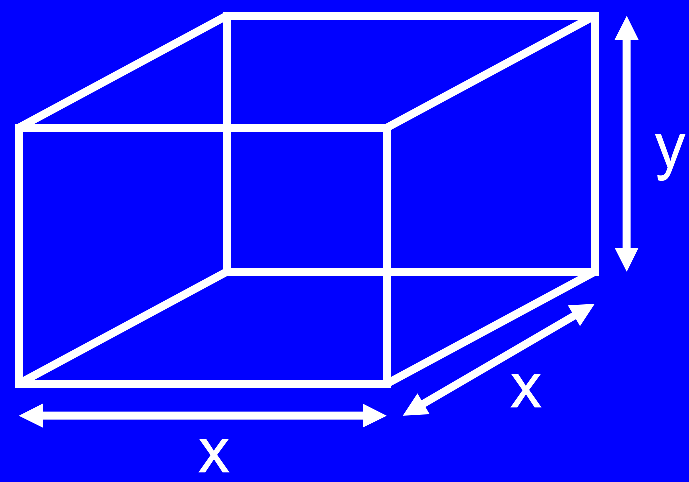
  base side x cm, height y cm (to choose)
- Parameter

  surface area bound $500\ cm^2$
- Constraint

  $x^2 + 4xy \leq 500$
- Objective

  maximize x^2y
- Optimization problem

$$
\begin{array}{ll}
\text{maximize} & x^2y \\
\text{subject to} & x^2 + 4xy \leq 500 \\
 & x, y \geq 0.
\end{array}
$$



Using Excel to Solve Optimization Problems

- Enter estimates for x and y in cells A1 and A2

  For example, enter 1 in cells A1 and A2

- Enter formulas for objective x^2y

  and constraint $x^2 + 4xy \leq 500$

  “= A1 ^ 2 * A2” in cell B1

  “= A1 ^ 2 + 4 * A1 * A2” in cell B2

	A	B
1	1	=A1^2*A2
2	1	=A1^2+4*A1*A2
3		
4		
5		
6		
7		

Using Excel to Solve Optimization Problems

- From “Tools” menu, select “Solver” and enter the following

  Objective: “Target Cell” is B1 and “Equal To” is “Max”

  Variables: “By Changing Cells” $A$1:$A$2

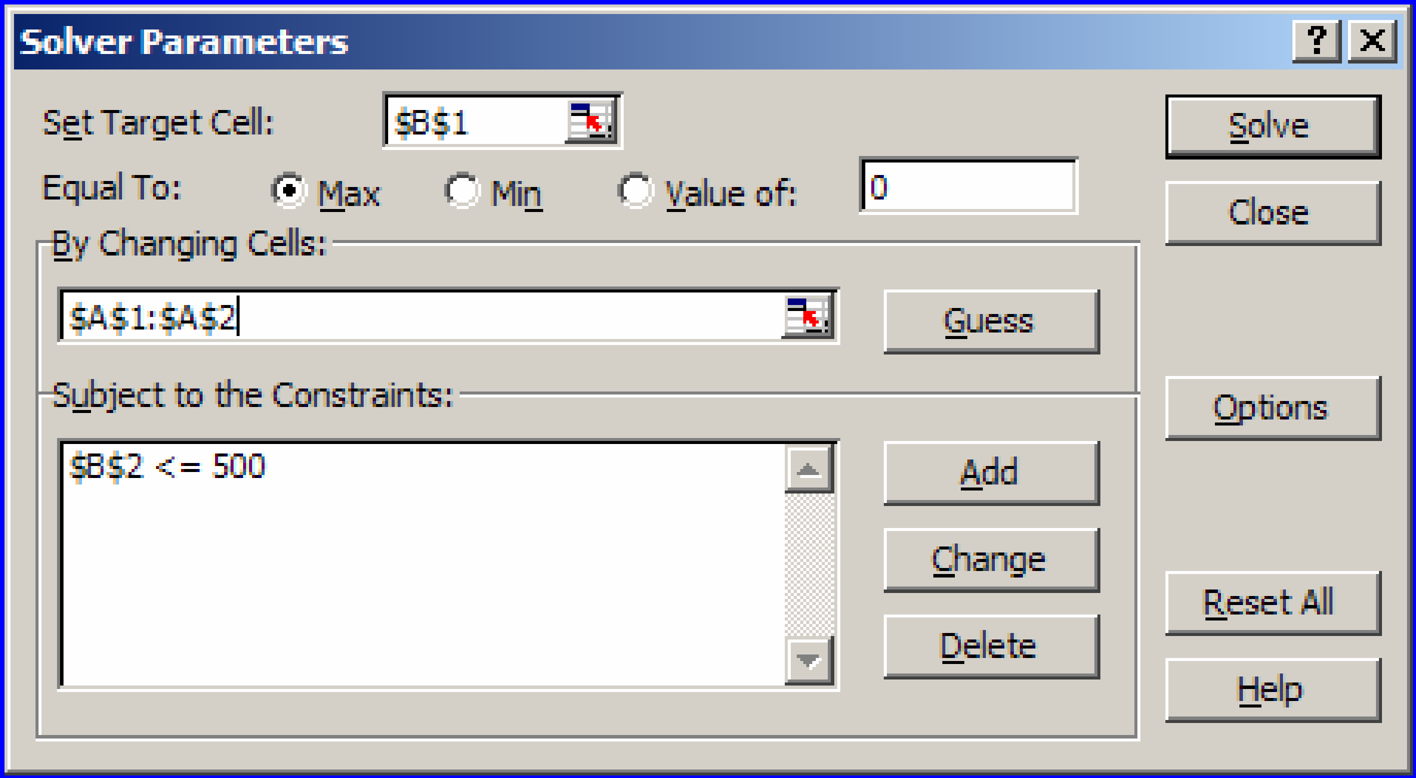
  Constraint: “Subject to Constraints”

  “Cell Reference” $B$2 is <= “Constraint” 500

  Click Options > Assume Nonnegative

  Clicking “Solve” starts the solver

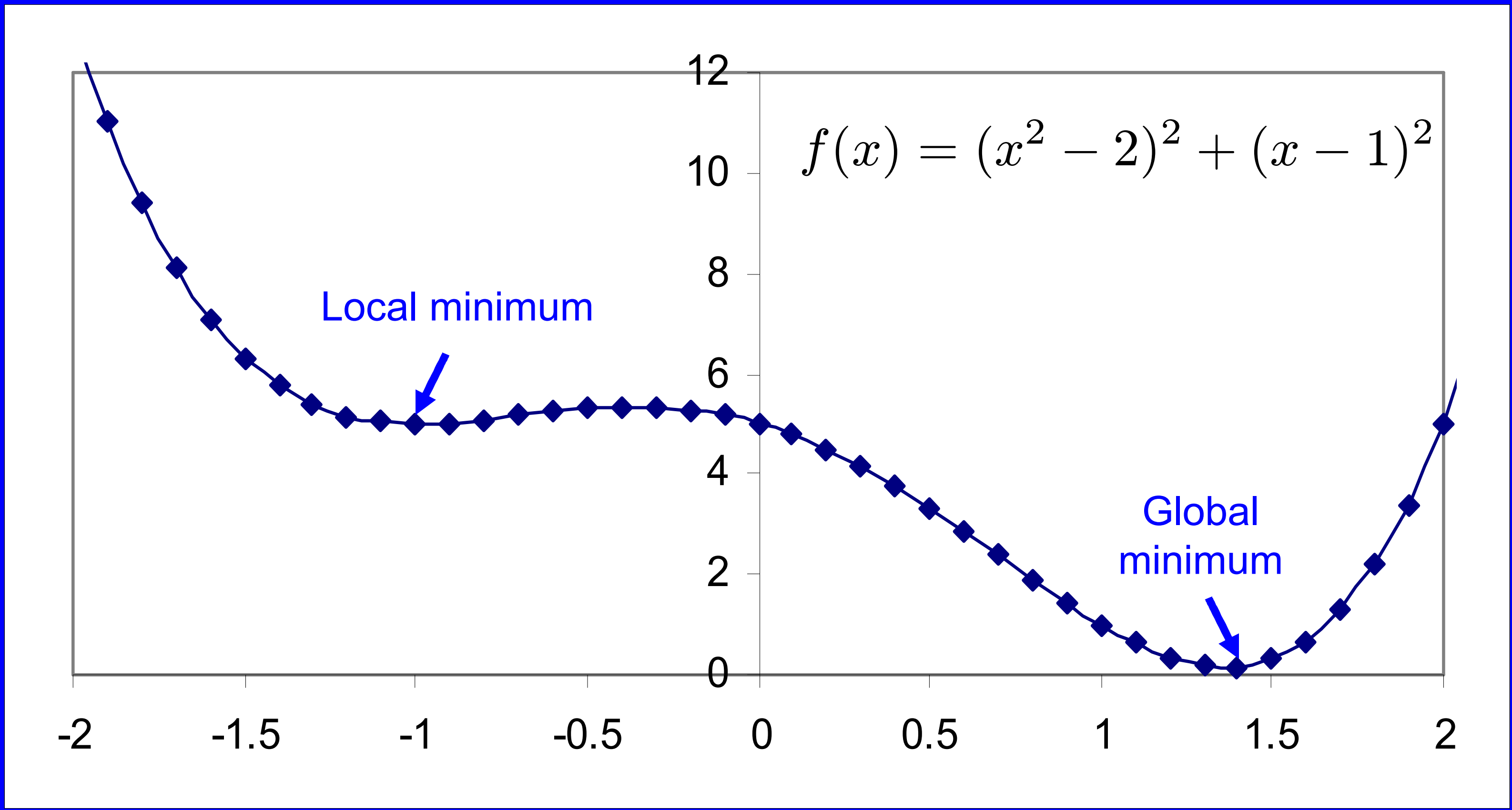
Using Excel to Solve Optimization Problems



Checking the Answer

- In our example, Excel's Solver adjusts the cells to
  base side x = A1 = 12.91 cm
  height y = A2 = 6.45 cm
  surface area = B2 = 500 cm2
  volume = B1 = 1075 cm3
- Questions:
  - Should we trust Solver's answer?
    (Caution! Try the Solver using the starting estimates x = 0 and y = 0)
  - How does the Solver work? (We care because it can fail!)
  - What else can we learn from Solver's answer?

Local and Global Solutions



Local and Global Solutions

- Ideally, we want a global solution x* such that

$$f(x^*) \leq f(x) \text{ for all } x$$

- Usually, solvers return a local solution x' such that

$$f(x') \leq f(x) \text{ for all } x \text{ near } x'$$

- Try minimizing $f(x) = (x^2-2)^2+(x-1)^2$ with Excel Solver

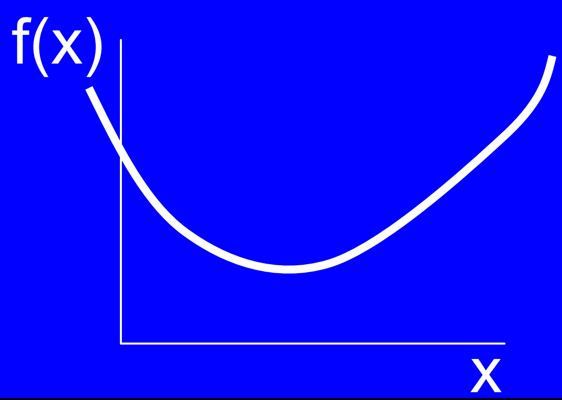
    Use an initial estimate of -2

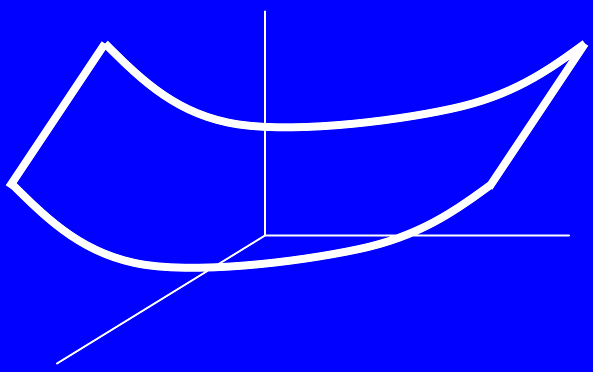
    Use an initial estimate of 3

- Bottom line is practical optimizers like Excel Solver find local solutions

Convexity

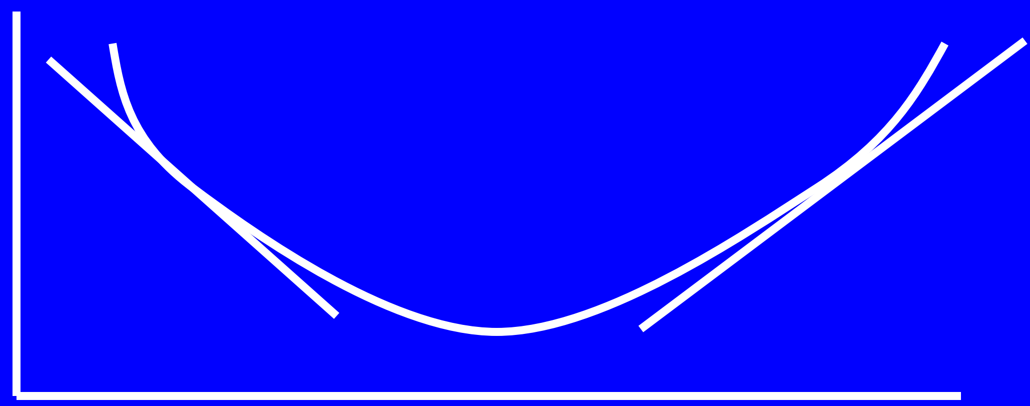
- If the objective function is convex, then any local solution is a global solution
- In this case, we do not need to worry about this local/global solution issue
- Some important objective functions are convex, but many are not!
- Why is Excel not able to reach a global solution?



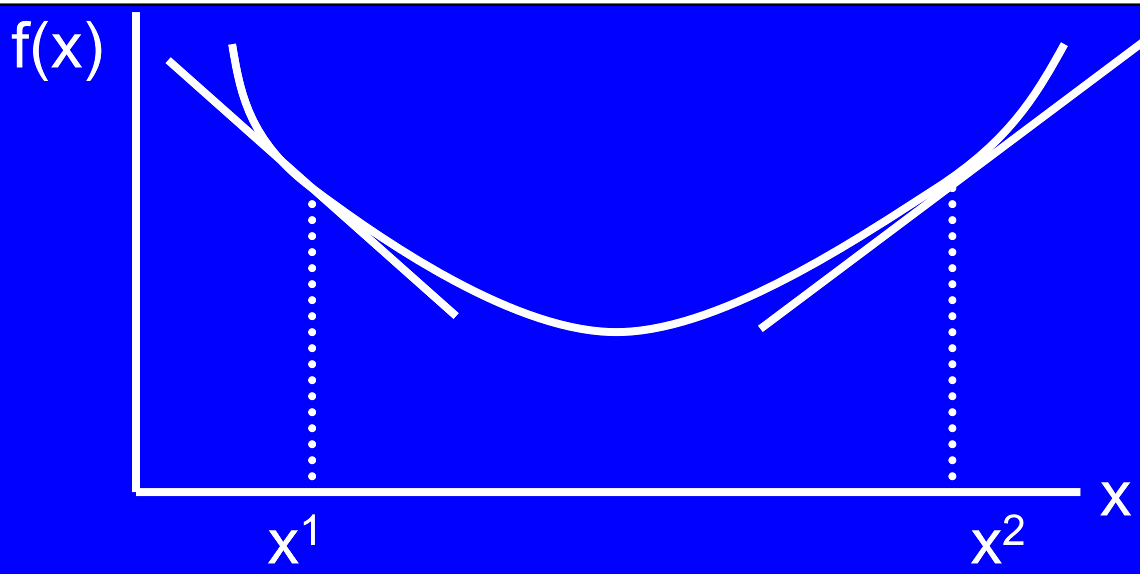


How Does Excel Solver Work?

- For a single dimensional function f(x), the derivative of f(x) points towards the direction of steepest ascent
- To minimize f(x), we should move in the opposite direction of the derivative of f(x)
- To maximize f(x), we should move in the same direction as the derivative of f(x)



How Does Excel Solver Work?



- Since $f'(x^1)<0$, if we move in the negative direction starting from x^1, then we increase the function's value
- Since $f'(x^2)>0$, if we move in the positive direction starting from x^2, then we increase the function's value

How Does Excel Solver Work?

- If the derivative is negative, then move in the negative direction to increase the function value
- If the derivative is positive, then move in the positive direction to increase the function value

- If the derivative is negative, then move in the positive direction to decrease the function value
- If the derivative is positive, then move in the negative direction to decrease the function value

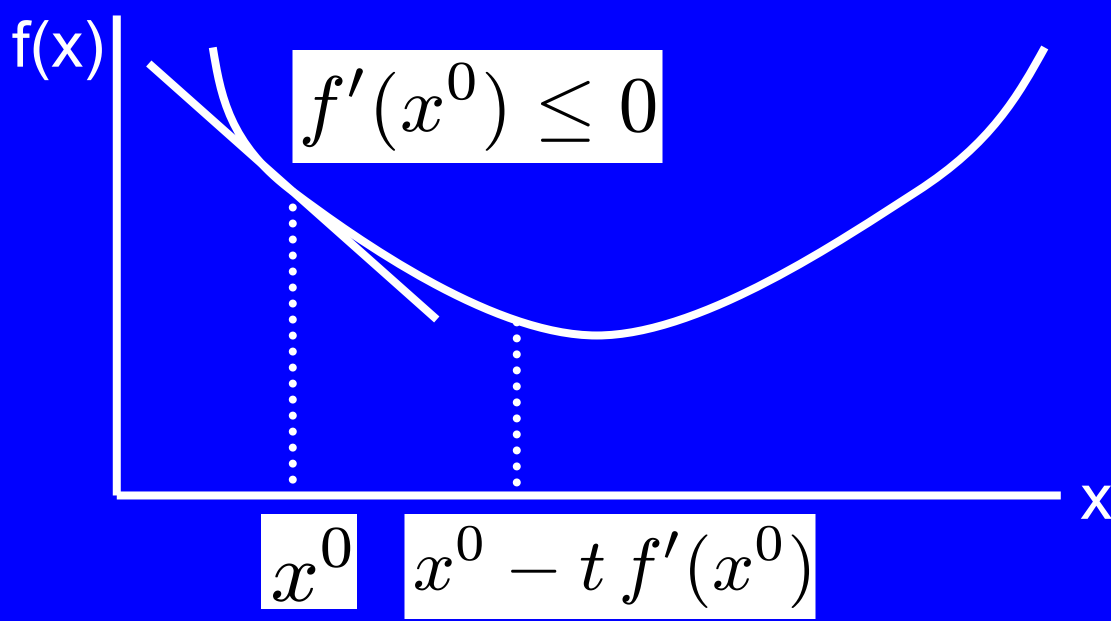
How Does Excel Solver Work?

- Assume that we are at x^0
- Compute the derivative of f(x) at x^0
- Move towards the direction $-f'(x^0)$ for a step size t
- This takes us to the point

$$\boxed{x^0 - t\, f'(x^0)}$$

If t is small enough, then we should obtain a decrease in the function value

How Does Excel Solver Work?



How Does Excel Solver Work?

- For single dimensional problems, all Excel solver does is to generate a sequence of solutions x^0, x^1, x^2,… by

$$\underbrace{x^{k+1}}_{\text{next point}} = \underbrace{x^k}_{\text{current point}} - \underbrace{t^k}_{\text{current step size}} \underbrace{f'(x^k)}_{\text{current derivative}}$$

How Does Excel Solver Work?

- To pick a step size, choose t^k arbitrarily and check if we have

$$f(x^{k+1}) = f(x^k - t^k\, f'(x^k)) \leq f(x^k)$$

- If not, then halve t^k and check again
- If t^k gets too small, then stop
- Usually works but slow
- Since this uses only local information, we can only get to a locally optimal solution

Extensions to Multidimensional Functions

- In multi-dimensional setting, the gradient of the function acts similar to the derivative
- Assume that we are dealing with a 3-dimensional function f(x,y,z)
- The gradient of this function is

$$\nabla f(x,y,z) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right]$$

- This vector points towards the direction of steepest ascent

Extensions to Multidimensional Functions

- To minimize f(x,y,z), the idea is to general a sequence of points (x^0,y^0,z^0), (x^1,y^1,z^1), (x^2,y^2,z^2),… by

gradient at current point

$$\left(x^{k+1}, y^{k+1}, z^{k+1}\right) = \left(x^k, y^k, z^k\right) - t^k \,\nabla f\left(x^k, y^k, z^k\right)$$

$$= \left(x^k, y^k, z^k\right) - t^k \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]_{\left(x^k, y^k, z^k\right)}$$

next point current point current step size

Extensions to Multidimensional Functions

- To pick a step size, choose t^k arbitrarily and check if we have

$$f(x^{k+1}, y^{k+1}, z^{k+1}) < f(x^k, y^k, z^k)$$

- If not, then halve t^k and try again
- If t^k gets too small, then stop
- Usually works but slow
- Since this uses only local information, we can only get to a locally optimal solution

Example

- We want to maximize

$$f(x,y) = \frac{500xy - 2x^2y^2}{x + 2y}$$

subject to the constraint that x ≥ 0, y ≥ 0

- Start with the point (x,y) = (10,5) with the current objective value f(10,5) = 1000
- Compute the gradient of f(x,y)

$$\frac{\partial f(x,y)}{\partial x} = \frac{(500y - 4xy^2)(x + 2y) - (500xy - 2x^2y^2)}{(x + 2y)^2}$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{(500x - 4x^2y)(x + 2y) - 2(500xy - 2x^2y^2)}{(x + 2y)^2}$$

Example

Microsoft Excel - optimization 1.xls

k	x	y	f(x,y)	Gr_x f(x,y)	Gr_y f(x,y)	step size	next point x	next point y	f(x_next, y_Next)
0	10.00	5.00	1000.00	25.00	50.00	1.00	35.00	55.00	-44474.14
0	10.00	5.00	1000.00	25.00	50.00	0.05	11.25	7.50	1064.73
1	11.25	7.50	1064.73	5.87	-11.48	0.10	11.84	6.35	1071.17
2	11.84	6.35	1071.17	7.92	8.81	0.20	13.42	8.11	1036.52
2	11.84	6.35	1071.17	7.92	8.81	0.10	12.63	7.23	1069.75
2	11.84	6.35	1071.17	7.92	8.81	0.05	12.23	6.79	1074.35
3	12.23	6.79	1074.35	2.49	-3.80	0.10	12.48	6.41	1075.09
4	12.48	6.41	1075.09	3.09	3.73	0.20	13.10	7.16	1068.69
4	12.48	6.41	1075.09	3.09	3.73	0.10	12.79	6.79	1074.67
4	12.48	6.41	1075.09	3.09	3.73	0.05	12.64	6.60	1075.57
5	12.64	6.60	1075.57	0.88	-1.86	0.10	12.72	6.41	1075.65

More on Step Size Choice

- Assume we want to minimize f(x,y,z)
- We are at point (x^k, y^k, z^k) and we have computed the gradient at (x^k, y^k, z^k)

$$\left(\nabla_x f(x^k, y^k, z^k), \nabla_y f(x^k, y^k, z^k), \nabla_z f(x^k, y^k, z^k)\right)$$

- If we move a t step size in the direction of this gradient we reach the point

$$x^k - t\,\nabla_x f(x^k, y^k, z^k),\;\; y^k - t\,\nabla_y f(x^k, y^k, z^k),\;\; z^k - t\,\nabla_z f(x^k, y^k, z^k)$$

- We are thinking of what step size t we should choose
- We could solve the optimization problem

$$\min_{t \geq 0} f(x^k - t\,\nabla_x f(x^k, y^k, z^k),\;\; y^k - t\,\nabla_y f(x^k, y^k, z^k),\;\; z^k - t\,\nabla_z f(x^k, y^k, z^k))$$

More on Step Size Choice

- This is an optimization problem with one variable
- Bottom line is if we can solve optimization problems with one variable, then we may be able to solve optimization problems with multiple variables