Network Revenue Management Models with Cancellations and Overbooking

1. Linear programming formulation with cancellations and overbooking

We begin by presenting a linear programming formulation of the network revenue management problem with cancellations and overbooking. After this, we use the Lagrangian relaxation strategy to show that this linear program is indeed a legitimate approximation of the network revenue management problem with cancellations and overbooking.

We use the same notation that we used earlier for the network revenue management problem without overbooking and cancellations. The problem takes place over the time periods $\{1, \ldots \tau\}$. All flights depart at time period $\tau + 1$. We have m flights and n itineraries. The capacity on flight leg i is c_i. If we accept a request for itinerary j, then we generate a revenue of f_j and consume a_{ij} units of capacity on flight leg i. If flight leg i is not in itinerary j, then we have $a_{ij} = 0$. As before, we assume that exactly one itinerary request arrives at each time period and the probability that we have a request for itinerary j at time period t is p_{jt}. Since exactly one itinerary request arrives at each time period, we have $\sum_{j=1}^{n} p_{jt} = 1$.

We assume that the cancellations for different reservations and the cancellations for the same reservation at different time periods are independent. Furthermore, we assume that the cancellations occur after the itinerary acceptance decisions at a particular time period. We let ρ_{jt} be the probability that we retain a reservation for itinerary j from time period t to time period $t+1$. With this definition, the probability that a reservation for itinerary j is cancelled at time period t is $1 - \rho_{jt}$. Furthermore, given that we have a reservation for itinerary j at time period t, the probability that we retain this reservation until the departure time is

Clearly, R_{jt} satisfies the recursion

Similarly, given that we have a reservation for itinerary j at time period t, the probability that we lose this reservation at some point before the departure time is

Clearly, L_{jt} satisfies the recursion

We let θ_j be the penalty cost of denying a reservation for itinerary j at the departure time and r_j be the refund if a reservation for itinerary j is cancelled before the departure time.

To formulate a linear program, we let w_{jt} be the number of reservations for itinerary j that we plan to accept at time period t and u_j be the number of reservations for itinerary j that we plan to deny boarding. Since R_{jt} is the probability of retaining a reservation for itinerary

j at time period t until the departure time, the expected number of reservations for itinerary j that we plan to retain until the departure time is

These are the number of reservations that are expected to show up at the departure time. In this case, we can write the capacity availability constraint as

On the other hand, if we accept a reservation for itinerary j at time period t, then we make a revenue of f_j, but we loose a revenue of r_j later on with probability L_{jt}. Therefore, the net revenue from accepting a reservation for itinerary j at time period t is $f_j - L_{jt}\, r_j$.

Putting it all together, we obtain the linear program

We note that the number of decision variables in this linear program is many more than the number of decisions variables in the linear program that does not incorporate cancellations.

The linear program above incorporates cancellations. An interesting question is whether this linear program incorporates no-shows. The answer to this question depends on how one interprets the retaining probabilities $\{\rho_{j\tau} : j = 1, \ldots, n\}$ at the last time period.

Using this linear program, making the acceptance decisions is easy. Assume that we have s_{jt} reservations for itinerary j at time period t. That is, the state of our reservation system is $\{s_{jt} : j = 1, \ldots, n\}$. In this case, the linear program that we solve is

Assume that we have a request for itinerary j^* at time period t. Letting $\{\mu_i : i = 1, \ldots, m\}$ be the optimal values of the dual variables associated with the first set of constraints, the decision rule we use is

We shortly elaborate on this decision rule a bit more.

2. Dynamic programming formulation with cancellations and overbooking

We let s_{jt} be the number of reservations that we have for itinerary j at the time period t. These reservations are counted at the beginning of the time period before we make the acceptance decisions or before the cancellations occur. In this case, the vector $s_t = \{s_{jt} : j = 1, \ldots, n\}$ gives the state of the system at time period t.

Given that we have b_{jt} reservations for itinerary j at time period t after the acceptance decisions, we let $Z_{jt}(b_{jt})$ be the number of reservations that we retain from time period t to time period $t + 1$. Due to our earlier independence assumptions, $Z_{jt}(b_{jt})$ has a binomial distribution with parameters

We use $Z_t(b_t)$ to denote the vector $\{Z_{jt}(b_{jt}) : j = 1, \ldots, n\}$.

Given that we reach the departure time with state $s_{\tau+1} = \{s_{j,\tau+1} : j = 1, \ldots, n\}$, we solve the problem

to decide which reservations should be denied boarding. This problem gives the terminal value function. Given that the state of the reservation system at time period t is s_t and we have a reservation request for itinerary j, the expected revenue obtained over the time periods $t, \ldots, \tau + 1$ satisfies the optimality equation where e_j is the unit vector with a 1 in the j-th component and the second term in the objective function above captures the expected refunds for the cancellations. We shortly clarify the boundary condition for the optimality equation.

Since we have $\mathbb{E}\{Z_{jt}(b_{jt})\} = \rho_{jt}\, b_{jt}$, we can write the optimality equation as

The inner summation on the right side above only depends on z and letting

the optimality equation becomes

$$\begin{aligned} V_t(s_t, j) &= \max_{y_{jt} \in \{0,1\}} \big[f_j - r_j\,(1-\rho_{jt})\big]\, y_{jt} - \sum_{j'=1}^{n} r_{j'}\,(1-\rho_{j't})\, s_{j't} \\ &\qquad\qquad + \sum_{z \in \mathbb{Z}_+^n} \mathbb{P}\{Z_t(s_t + y_{jt}\, e_j) = z\}\, \bar{V}_{t+1}(z) \\ &= \max_{y_{jt} \in \{0,1\}} \big[f_j - r_j\,(1-\rho_{jt})\big]\, y_{jt} - \sum_{j'=1}^{n} r_{j'}\,(1-\rho_{j't})\, s_{j't} + \mathbb{E}_{Z_t}\{\bar{V}_{t+1}(Z_t(s_t + y_{jt}\, e_j))\}. \end{aligned}$$

Taking the expectations of both sides over the itinerary request j, we obtain

We write the final optimality equation as

with the boundary condition $\bar{V}_{\tau+1}(s_{\tau+1})$ as above.

3. Connections between the dynamic programming formulation and the linear programming approximation

Consider dropping the capacity availability constraints in the problem that we solve at the departure time, which is given by

$$\begin{aligned}
\bar{V}_{\tau+1}(s_{\tau+1}) = \max \quad & -\sum_{j=1}^{n} \theta_j \, u_j \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij} \, (s_{j,\tau+1} - u_j) \le c_i \qquad i = 1, \ldots, m \\
& 0 \le u_j \le s_{j,\tau+1} \qquad\qquad\qquad j = 1, \ldots, n.
\end{aligned}$$

Furthermore, consider associating the positive Lagrange multipliers $\{\lambda_i : i = 1, \ldots, m\}$ with the capacity availability constraints at the departure time to add these constraints to the objective function. In particular, we solve the problem

for the departure time, whereas we continue solving the optimality equation

$$\bar{V}_t^{\lambda}(s_t) = \max_{y_t \in \{0,1\}^n} \sum_{j=1}^{n} p_{jt} \left\{ \left[f_j - r_j \, (1 - \rho_{jt}) \right] y_{jt} + \mathbb{E}\{ \bar{V}_{t+1}^{\lambda}(Z_t(s_t + y_{jt} \, e_j)) \} \right\} - \sum_{j=1}^{n} r_j \, (1 - \rho_{jt}) \, s_{jt}.$$

We use the superscript λ in the value functions to emphasize that the solution to the optimality equation above depends on the Lagrange multipliers.

Similar to our earlier results, it can be shown that the Lagrangian relaxation strategy gives an upper bound on the value functions. That is, we have $\bar{V}_t(s_t) \le \bar{V}_t^{\lambda}(s_t)$ for all time periods and for all reservation levels, as long as the Lagrange multipliers are positive.

Another property of the Lagrangian relaxation strategy is that for a fixed value of the Lagrange multipliers, the value functions $\{\bar{V}_t^{\lambda}(\cdot) : t = 1, \ldots, \tau\}$ can easily be computed. In particular, we have

$$\bar{V}_t^{\lambda}(s_t) = \sum_{i=1}^{m} \lambda_i \, c_i + \sum_{j=1}^{n} [R_{jt} \, v_j^{\lambda} - L_{jt} \, r_j] \, s_{jt} + \sum_{t'=t}^{\tau} \sum_{j=1}^{n} p_{jt'} \, \left[f_j - L_{jt'} \, r_j + R_{jt'} \, v_j^{\lambda} \right]^+ ,$$

where we let $v_j^{\lambda} = -\min\{\sum_{i=1}^{n} a_{ij} \, \lambda_i, \theta_j\}$. This result is easy to show by induction. Consider the problem we solve at the departure time. We have

Therefore, we have $\bar{V}_{\tau+1}^{\lambda}(s_t) = \sum_{i=1}^{m} \lambda_i \, c_i + \sum_{j=1}^{n} v_j^{\lambda} \, s_{j,\tau+1}$. Consider the problem that we solve at time period τ. We have

Therefore, the result holds for time period τ. Assuming that the result holds for time period $t+1$, we have

This completes the proof.

Similar to our earlier results, since the system starts with no reservations and we have $\bar{V}_1(0) \le V_1^\lambda(0)$, we can solve the problem

$$\min_{\lambda \ge 0} V_1^\lambda(0)$$

to obtain a good set of Lagrange multipliers. As complicated as it looks, the minimization problem above can be solved as a linear program.

To see this, we first note that for a fixed value of λ, we have

$$\bar{V}_1^\lambda(0) = \sum_{i=1}^{m} \lambda_i \, c_i + \sum_{t=1}^{\tau} \sum_{j=1}^{n} p_{jt} \, \left[f_j - L_{jt} \, r_j + R_{jt} \, v_j^\lambda \right]^+ .$$

The first thing to note is that for a fixed value of λ, $\bar{V}_1^\lambda(0)$ is the optimal solution to the trivial linear program

If we want to solve the problem $\min_{\lambda \geq 0} V_1^\lambda(0)$, then all we need to do is to treat the Lagrange multipliers in the linear program above as decision variables and solve the problem

It turns out that solving the linear program above, or equivalently solving the problem $\min_{\lambda \geq 0} V_1^{\lambda}(0)$, is equivalent to solving the deterministic linear program that we formulated at the very beginning. To see this, the dual of the last linear program is