Network Revenue Management Models with Cancellations and Overbooking

1. Linear programming formulation with cancellations and overbooking

We begin by presenting a linear programming formulation of the network revenue management problem with cancellations and overbooking. After this, we use the Lagrangian relaxation strategy to show that this linear program is indeed a legitimate approximation of the network revenue management problem with cancellations and overbooking.

We use the same notation that we used earlier for the network revenue management problem without overbooking and cancellations. The problem takes place over the time periods $\{1, \ldots \tau\}$. All flights depart at time period $\tau + 1$. We have m flights and n itineraries. The capacity on flight leg i is c_i. If we accept a request for itinerary j, then we generate a revenue of f_j and consume a_{ij} units of capacity on flight leg i. If flight leg i is not in itinerary j, then we have $a_{ij} = 0$. As before, we assume that exactly one itinerary request arrives at each time period and the probability that we have a request for itinerary j at time period t is p_{jt}. Since exactly one itinerary request arrives at each time period, we have $\sum_{j=1}^{n} p_{jt} = 1$.

We assume that the cancellations for different reservations and the cancellations for the same reservation at different time periods are independent. Furthermore, we assume that the cancellations occur after the itinerary acceptance decisions at a particular time period. We let ρ_{jt} be the probability that we retain a reservation for itinerary j from time period t to time period $t+1$. With this definition, the probability that a reservation for itinerary j is cancelled at time period t is $1 - \rho_{jt}$. Furthermore, given that we have a reservation for itinerary j at time period t, the probability that we retain this reservation until the departure time is

$$R_{jt} = q_{jt} \; q_{jt+1} \cdots q_{jT}$$

Clearly, R_{jt} satisfies the recursion

$$R_{jt} = q_{jt} \cdot R_{jt+1}$$

Similarly, given that we have a reservation for itinerary j at time period t, the probability that we lose this reservation at some point before the departure time is

$$L_{jt} = (1 - q_{jt}) + q_{jt} (1 - q_{jt+1}) + q_{jt} \, q_{jt+1} (1 - q_{jt+2}) + \cdots$$

Clearly, L_{jt} satisfies the recursion

$$L_{jt} = (1 - q_{jt}) + q_{jt} \cdot L_{jt+1}$$

We let θ_j be the penalty cost of denying a reservation for itinerary j at the departure time and r_j be the refund if a reservation for itinerary j is cancelled before the departure time.

To formulate a linear program, we let w_{jt} be the number of reservations for itinerary j that we plan to accept at time period t and u_j be the number of reservations for itinerary j that we plan to deny boarding. Since R_{jt} is the probability of retaining a reservation for itinerary

j at time period t until the departure time, the expected number of reservations for itinerary j that we plan to retain until the departure time is

$$\sum_{t=1}^{T} \rho_{jt} \, w_{jt}$$

These are the number of reservations that are expected to show up at the departure time. In this case, we can write the capacity availability constraint as

$$\sum_{j=1}^{n} a_{ij} \sum_{t=1}^{T} \rho_{jt} \, w_{jt} - \sum_{j=1}^{n} a_{ij} \, u_j \le c_i$$

On the other hand, if we accept a reservation for itinerary j at time period t, then we make a revenue of f_j, but we loose a revenue of r_j later on with probability L_{jt}. Therefore, the net revenue from accepting a reservation for itinerary j at time period t is $f_j - L_{jt} \, r_j$.

Putting it all together, we obtain the linear program

$$\max \quad \sum_{j=1}^{n} \sum_{t=1}^{T} (f_j - L_{jt} \, r_j) \, w_{jt} - \sum_{j=1}^{n} \theta_j \, u_j$$

$$\text{st} \quad \sum_{j} \sum_{t} a_{ij} \, \rho_{jt} \, w_{jt} - \sum_{j} a_{ij} \, u_j \le c_i \qquad i = 1, \ldots, m$$

$$0 \le w_{jt} \le \rho_{jt} \qquad j = 1, \ldots, n, \; t = 1, \ldots, T$$

$$0 \le u_j \le \sum_{t} \rho_{jt} \, w_j \qquad j = 1, \ldots, n$$

We note that the number of decision variables in this linear program is many more than the number of decisions variables in the linear program that does not incorporate cancellations.

The linear program above incorporates cancellations. An interesting question is whether this linear program incorporates no-shows. The answer to this question depends on how one interprets the retaining probabilities $\{\rho_{j\tau} : j = 1, \ldots, n\}$ at the last time period.

We can interpret ρ_{jT} as no show + cancellation probabilities

Using this linear program, making the acceptance decisions is easy. Assume that we have s_{jt} reservations for itinerary j at time period t. That is, the state of our reservation system is $\{s_{jt} : j = 1, \ldots, n\}$. In this case, the linear program that we solve is

$$\max \quad \sum_{t'=t}^{T} \sum_{j=1}^{n} (f_j - L_{jt} \, r_j) \, w_{jt} - \sum \theta_j \, u_j$$

$$\text{st} \quad \sum_{j} \sum_{t'=t}^{T} a_{ij} \, \rho_{jt'} \, w_{jt'} + \sum_{j=1}^{n} a_{ij} \, \rho_{jt} \, s_{jt} - \sum_{j} a_{ij} \, u_j \le c_i \qquad i = 1, \ldots, n$$

$$0 \le w_{jt'} \le \rho_{jt'} \qquad j = 1, \ldots, n \quad 2 \; t' = t, \ldots, T$$

$$0 \le u_j \le \sum_{t'=t}^{T} \rho_{jt} \, w_j + \rho_{jt} \, s_{jt} \qquad j = 1, \ldots, n$$

Assume that we have a request for itinerary j^* at time period t. Letting $\{\mu_i : i = 1, \ldots, m\}$ be the optimal values of the dual variables associated with the first set of constraints, the decision rule we use is

$$\text{Accept if} \quad f_{j^*} - L_{j^*t}\, r_{j^*} \geq R_{j^*t} \left(\sum a_{ij^*} \mu_i \right)$$

$$\text{OR} \quad f_{j^*} - L_{j^*t}\, r_{j^*} \geq R_{j^*t} \cdot \theta_{j^*}$$

We shortly elaborate on this decision rule a bit more.

2. Dynamic programming formulation with cancellations and overbooking

We let s_{jt} be the number of reservations that we have for itinerary j at the time period t. These reservations are counted at the beginning of the time period before we make the acceptance decisions or before the cancellations occur. In this case, the vector $s_t = \{s_{jt} : j = 1, \ldots, n\}$ gives the state of the system at time period t.

Given that we have b_{jt} reservations for itinerary j at time period t after the acceptance decisions, we let $Z_{jt}(b_{jt})$ be the number of reservations that we retain from time period t to time period $t+1$. Due to our earlier independence assumptions, $Z_{jt}(b_{jt})$ has a binomial distribution with parameters

$$b_{jt} \text{ and } \rho_{jt}$$

We use $Z_t(b_t)$ to denote the vector $\{Z_{jt}(b_{jt}) : j = 1, \ldots, n\}$.

Given that we reach the departure time with state $s_{\tau+1} = \{s_{j,\tau+1} : j = 1, \ldots, n\}$, we solve the problem

$$V_{\tau+1}(s_{j,\tau+1}) = \max \; - \sum \theta_j u_j$$

$$\text{st} \quad \sum a_{ij} (s_{j,\tau+1} - u_j) \leq c_i$$

$$0 \leq u_j \leq s_{j,\tau+1}$$

to decide which reservations should be denied boarding. This problem gives the terminal value function. Given that the state of the reservation system at time period t is s_t and we have a reservation request for itinerary j, the expected revenue obtained over the time periods $t, \ldots, \tau+1$ satisfies the optimality equation

$$V_t(s_t, j) = \max_{y_{jt} \in \{0,1\}} f_j y_{jt} - \underbrace{\sum_{j'=1}^{n} r_{j'} \cdot \mathbb{E}\{ s_{j't} - Z_{j't}(s_{j't}) \}} + \sum_{z \in Z_+^n} \sum_{j'=1}^{n} V_{t+1}(z, j') \cdot p_{j',t+1} \, \mathbb{P}\{ Z_t(s_t + y_{jt} e_j) = z \}$$

$$\underbrace{\sum_{j'=1}^{n} r_{j'} \, \mathbb{E}\{ s_{j't} + \mathbb{1}(j'=j)\, y_{j't} - Z_{j't}(s_{j't} + \mathbb{1}(j'=j)\, y_{j't}) \}}$$

$$\sum_{j'=1}^{n} r_{j'} \left(s_{j't} + \mathbb{1}(j'=j)\, y_{j't} - \rho_{j't} \left(s_{j't} + \mathbb{1}(j'=j)\, y_{j't} \right) \right)$$

where e_j is the unit vector with a 1 in the j-th component and the second term in the objective function above captures the expected refunds for the cancellations. We shortly clarify the boundary condition for the optimality equation.

Since we have $\mathbb{E}\{Z_{jt}(b_{jt})\} = \rho_{jt}\, b_{jt}$, we can write the optimality equation as

$$V_t(s_t, j) = \max_{y_{jt}\in\{0,1\}} \left\{ (f_j - r_j + \rho_{jt} r_j)\, y_{jt} + \sum_{z \in \mathbb{Z}_+^n} \sum_{j'=1}^{n} V_{t+1}(z, j')\, p_{j't+1} \cdot \mathbb{P}\{Z_t(s_t + y_{jt} e_j) = z\} \right\}$$
$$- \sum_{j'=1}^{n} r_{j'}\,(1-\rho_{j't})\, s_{j't}$$

The inner summation on the right side above only depends on z and letting

$$\sum_{j'=1}^{n} V_{t+1}(z, j')\, p_{j't+1} = \bar{V}_{t+1}(z)$$

the optimality equation becomes

$$\begin{aligned} V_t(s_t, j) &= \max_{y_{jt}\in\{0,1\}} \left[f_j - r_j\,(1-\rho_{jt})\right] y_{jt} - \sum_{j'=1}^{n} r_{j'}\,(1-\rho_{j't})\, s_{j't} \\ &\qquad\qquad + \sum_{z\in\mathbb{Z}_+^n} \mathbb{P}\{Z_t(s_t + y_{jt}\, e_j) = z\}\, \bar{V}_{t+1}(z) \\ &= \max_{y_{jt}\in\{0,1\}} \left[f_j - r_j\,(1-\rho_{jt})\right] y_{jt} - \sum_{j'=1}^{n} r_{j'}\,(1-\rho_{j't})\, s_{j't} + \mathbb{E}_{Z_t}\{\bar{V}_{t+1}(Z_t(s_t + y_{jt}\, e_j))\}. \end{aligned}$$

Taking the expectations of both sides over the itinerary request j, we obtain

$$\bar{V}_t(s_t) = \sum_{j=1}^{n} p_{jt}\, V_t(s_t, j)$$
$$= \sum_{j=1}^{n} p_{jt} \left\{ \max_{y_{jt}\in\{0,1\}} \left(f_j - (1-\rho_{jt})\, r_j\right) y_{jt} + \mathbb{E}_{Z_t}\left\{\bar{V}_{t+1}(Z_t(s_t + y_{jt} e_j))\right\} \right\}$$
$$- \sum_{j'} r_{j'}\,(1-\rho_{j't})\, s_{j't}$$

We write the final optimality equation as

$$V_t(s_t) = \max_{y_t \in \{0,1\}^n} \Big\{ \sum p_{jt} \, (f_j - r_j(1-\rho_{jt}))\, y_{jt} + \mathbb{E}_{Z_t}\Big\{ V_{t+1}\big(Z_t\,(s_t + y_{jt}\, e_j)\big)\Big\} - \sum_{j'} r_{j'}\,(1-\rho_{j't})\, s_{j't} \Big\}$$

with the boundary condition $\bar{V}_{\tau+1}(s_{\tau+1})$ as above.

3. Connections between the dynamic programming formulation and the linear programming approximation

Consider dropping the capacity availability constraints in the problem that we solve at the departure time, which is given by

$$\begin{aligned} \bar{V}_{\tau+1}(s_{\tau+1}) = \max \quad & -\sum_{j=1}^{n} \theta_j \, u_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij}\,(s_{j,\tau+1} - u_j) \le c_i \qquad i = 1, \ldots, m \\ & 0 \le u_j \le s_{j,\tau+1} \qquad\qquad\quad j = 1, \ldots, n. \end{aligned}$$

Furthermore, consider associating the positive Lagrange multipliers $\{\lambda_i : i = 1, \ldots, m\}$ with the capacity availability constraints at the departure time to add these constraints to the objective function. In particular, we solve the problem

$$\begin{aligned} \bar{V}^{\lambda}_{T+1}(s_{T+1}) = \max \quad & -\sum_j \theta_j u_j + \sum_i \lambda_i c_i - \sum_i \sum_j \lambda_i a_{ij} (s_{j,T+1} - u_j) \\ \text{st} \quad & \\ & 0 \le u_j \le s_{j,T+1} \qquad j = 1, \ldots, n \end{aligned}$$

for the departure time, whereas we continue solving the optimality equation

$$\bar{V}^{\lambda}_t(s_t) = \max_{y_t \in \{0,1\}^n} \sum_{j=1}^{n} p_{jt} \left\{ \left[f_j - r_j\,(1-\rho_{jt}) \right] y_{jt} + \mathbb{E}\{\bar{V}^{\lambda}_{t+1}(Z_t(s_t + y_{jt}\, e_j))\} \right\} - \sum_{j=1}^{n} r_j\,(1-\rho_{jt})\, s_{jt}.$$

We use the superscript λ in the value functions to emphasize that the solution to the optimality equation above depends on the Lagrange multipliers.

Similar to our earlier results, it can be shown that the Lagrangian relaxation strategy gives an upper bound on the value functions. That is, we have $\bar{V}_t(s_t) \le \bar{V}^{\lambda}_t(s_t)$ for all time periods and for all reservation levels, as long as the Lagrange multipliers are positive.

Another property of the Lagrangian relaxation strategy is that for a fixed value of the Lagrange multipliers, the value functions $\{\bar{V}^{\lambda}_t(\cdot) : t = 1, \ldots, \tau\}$ can easily be computed. In particular, we have

$$\bar{V}_t^{\lambda}(s_t) = \sum_{i=1}^{m} \lambda_i \, c_i + \sum_{j=1}^{n} [R_{jt} \, v_j^{\lambda} - L_{jt} \, r_j] \, s_{jt} + \sum_{t'=t}^{\tau} \sum_{j=1}^{n} p_{jt'} \, \left[f_j - L_{jt'} \, r_j + R_{jt'} \, v_j^{\lambda}\right]^+ ,$$

where we let $v_j^{\lambda} = -\min\{\sum_{i=1}^{n} a_{ij} \, \lambda_i, \theta_j\}$. This result is easy to show by induction. Consider the problem we solve at the departure time. We have

$$\bar{V}_{\tau+1}^{\lambda}(s_{\tau+1}) = \max \sum_j \Big(\sum_i a_{ij}\lambda_i - \theta_j\Big) u_j - \sum_j \sum_i a_{ij} \lambda_i \, s_{j,\tau+1} + \sum_i \lambda_i c_i$$

st

$$0 \le u_j \le s_{j,\tau+1}$$

$$= \sum_j \Big(\sum_i a_{ij}\lambda_i - \theta_j\Big)^+ s_{j,\tau+1} - \sum_i \sum_j a_{ij} \lambda_i \, s_{j\tau+1} + \sum_i \lambda_i c_i$$

$$= \sum_j v_j^{\lambda} \, s_{j,\tau+1} + \sum_i \lambda_i c_i$$

Therefore, we have $\bar{V}_{\tau+1}^{\lambda}(s_t) = \sum_{i=1}^{m} \lambda_i \, c_i + \sum_{j=1}^{n} v_j^{\lambda} \, s_{j,\tau+1}$. Consider the problem that we solve at time period τ. We have

$$\bar{V}_{\tau}^{\lambda}(s_\tau) = \max \sum_j p_{j\tau} \Big\{ [f_j - r_j(1-\rho_{j\tau})] \, y_{j\tau} + \mathbb{E}\Big\{ \sum_{j'=1}^{n} v_{j'}^{\lambda} \cdot Z_{j'\tau} \big(s_{j'\tau} + y_{j\tau} \cdot \mathbb{1}(j=j')\big)\Big\}\Big\}$$

$$+ \sum \lambda_i c_i - \sum_j r_j (1-\rho_{j\tau}) \, s_{j\tau}$$

$$= \max \sum_j p_{j\tau} \Big\{ [f_j - r_j(1-\rho_{j\tau})] \, y_{j\tau} + \sum_{j'} v_{j'}^{\lambda} \cdot \rho_{j'\tau} \big(s_{j'\tau} + y_{j\tau} \, \mathbb{1}(j'=j)$$

$$+ \cdots$$

$$= \max \sum_j p_{j\tau} \Big\{ [f_j - r_j(1-\rho_{j\tau}) + \rho_{j\tau} \cdot v_j^{\lambda} \cdot] \, y_{j\tau} \Big\} + \sum_{j'} v_{j'}^{\lambda} \rho_{j'\tau} \, s_{j'\tau}$$

$$+ \sum \lambda_i c_i - \sum_j r_j (1-\rho_{j\tau}) \, s_{j\tau}$$

$$= \sum_j p_{j\tau} \left[f_j - r_j L_{j\tau} + R_{j\tau} v_j^{\lambda}\right]^+ + \sum_j [v_j^{\lambda} R_{j\tau} - r_j L_{j\tau}] \, s_{j\tau}$$

$$+ \sum_i \lambda_i c_i$$

Therefore, the result holds for time period τ. Assuming that the result holds for time period $t+1$, we have

$$\bar{V}_{t}^{\lambda}(s_t) = \max \sum_j p_{jt} \left\{ [f_j - r_j (1-\rho_{jt})] y_{jt} + \mathbb{E}\left\{ \sum_{j'=1}^{n} \left(R_{j't+1} v_{j'}^{\lambda} - L_{j't+1} r_{j'}\right) Z_{j't}(s_{j't} + y_{jt} \cdot \mathbb{1}(j=j')) \right\} \right.$$

$$+ \boxed{\lambda} + \boxed{[\quad]^+} - \sum_{j=1}^{n} r_j (1-\rho_{jt}) s_{jt}$$

$$= \max \sum_j p_{jt} \left\{ [f_j - r_j (1-\rho_{jt}) + \rho_{jt} R_{jt+1} v_j^{\lambda} - r_j L_{jt+1} \rho_{jt}] y_{jt} \right\}$$

$$+ \sum_j \left(R_{jt+1} v_j^{\lambda} - L_{jt+1} r_j\right) \rho_{jt} s_{jt} + \sum \lambda_i c_i + \sum_{t'=t+1}^{T} \sum p_{jt'} [\ldots]^+$$

$$- \sum_j r_j (1-\rho_{jt}) s_{jt}$$

$$= \sum_t \sum_j p_{jt} [f_j - r_j L_{jt} + R_{jt} v_j^{\lambda}]^+ + \sum_j [R_{jt} v_j^{\lambda} - L_{jt} r_j] s_{jt}$$

$$+ \sum_i \lambda_i c_i$$

This completes the proof.

Similar to our earlier results, since the system starts with no reservations and we have $\bar{V}_1(0) \le V_1^{\lambda}(0)$, we can solve the problem

$$\min_{\lambda \ge 0} V_1^{\lambda}(0)$$

to obtain a good set of Lagrange multipliers. As complicated as it looks, the minimization problem above can be solved as a linear program.

To see this, we first note that for a fixed value of λ, we have

$$\bar{V}_1^\lambda(0) = \sum_{i=1}^{m} \lambda_i c_i + \sum_{t=1}^{\tau} \sum_{j=1}^{n} p_{jt} \left[f_j - L_{jt} r_j + R_{jt} v_j^\lambda\right]^+ .$$

The first thing to note is that for a fixed value of λ, $\bar{V}_1^\lambda(0)$ is the optimal solution to the trivial linear program

$$\min \sum_{i=1}^{M} \lambda_i c_i + \sum_t \sum_j p_{jt} \beta_{jt}$$

st

$$\beta_{jt} \geq f_j - L_{jt} r_j - R_{jt} \min \left\{ \sum_i a_{ij} \lambda_i, \theta_j \right\}$$

$$\beta_{jt} \geq 0$$

$$\min \sum_i \lambda_i c_i + \sum_t \sum_j p_{jt} \beta_{jt}$$

st

$$\beta_{jt} + R_{jt} \sum_i a_{ij} \lambda_i \geq f_j - L_{jt} r_j \qquad j = 1, \dots, n \qquad t = 1, \dots T$$

$$\beta_{jt} + R_{jt} \theta_j \geq f_j - L_{jt} r_j \qquad j = 1, \dots, n \qquad t = 1, \dots T$$

$$\beta_{jt} \geq 0 \qquad j = 1, \dots, n \qquad t = 1, \dots T$$

If we want to solve the problem $\min_{\lambda \geq 0} V_1^\lambda(0)$, then all we need to do is to treat the Lagrange multipliers in the linear program above as decision variables and solve the problem

$$\min \sum_i \lambda_i c_i + \sum_t \sum_j p_{jt} \beta_{jt}$$

st

$$\beta_{jt} + \sum_i a_{ij} R_{jt} \lambda_i \geq f_j - L_{jt} r_j \qquad j = 1, \dots, n, \quad t = 1, \dots, T$$

$$\beta_{jt} \geq f_j - L_{jt} r_j - R_{jt} \theta_j \qquad j = 1, \dots, n, \quad t = 1, \dots, T$$

$$\beta_{jt} \geq 0 \qquad j = 1, \dots, n \qquad t = 1, \dots, T$$

$$\lambda_i \geq 0 \qquad i = 1, \dots, m$$

It turns out that solving the linear program above, or equivalently solving the problem $\min_{\lambda \geq 0} V_1^{\lambda}(0)$, is equivalent to solving the deterministic linear program that we formulated at the very beginning. To see this, the dual of the last linear program is

$$\max \quad \sum_j \sum_T (f_j - L_{jt} r_j) \, y_{jt} + \sum_j \sum_t (f_j - L_{jt} r_j - R_{jt} \theta_j) \, z_{jt}$$

st

$$\sum_t \sum_j a_{ij} R_{jt} y_{jt} \leq c_i \qquad i = 1 .. m$$

$$y_{jt} + z_{jt} \leq p_{jt} \qquad j = 1 .. n \qquad t = 1 .., T$$

$$y_{jt}, \quad z_{jt} \geq 0$$

$$y_{jt} + z_{jt} = w_{jt} \qquad \qquad \sum R_{jt} \, z_{jt} = u_j$$

$$\max \quad \sum_j \sum_t (f_j - L_{jt} r_j) \, w_{jt} - \sum \theta_j \, u_j$$

st

$$\sum_j \sum_t a_{ij} R_{jt} w_{jt} - \sum_j a_{ij} u_j \leq c_i \qquad i = 1 .. m$$

$$0 \leq w_{jt} \leq p_{jt} \qquad j = 1 .., n \qquad t = 1 .., T$$

$$u_j \leq \sum_t R_{jt} w_{jt} \qquad j = 1 ... n$$