Computing Time-Dependent Bid Prices in Network Revenue Management Problems

Sumit Kunnumkal Indian School of Business, Gachibowli, Hyderabad, 500032, India sumit_kunnumkal@isb.edu

Huseyin Topaloglu School of Operations Research and Information Engineering, Cornell University, Ithaca, New York 14853, USA topaloglu@orie.cornell.edu

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Abstract

We propose a new method to compute bid prices in network revenue management problems. The novel aspect of our method is that it naturally provides dynamic bid prices that depend on how much time is left until departure. We show that our method provides an upper bound on the optimal total expected revenue and this upper bound is tighter than the one provided by the widely known deterministic linear programming approach. Furthermore, it is possible to use the bid prices computed by our method as a starting point in a dynamic programming decomposition-like idea to decompose the network revenue management problem by the flight legs and to obtain dynamic and capacity-dependent bid prices. Our computational experiments indicate that the proposed method improves on many standard benchmarks.

The idea of bid prices forms a powerful tool for building good and practical policies for network revenue management problems. This idea associates a bid price with each flight leg that captures the opportunity cost of a unit of capacity. An itinerary request is accepted only when there is enough capacity and the revenue from the itinerary request exceeds the sum of the bid prices associated with the flight legs that are in the requested itinerary; see Williamson (1992) and Talluri and van Ryzin (1998). Although it is known that the optimal policies are not necessarily characterized by bid prices, the intuitive appeal and ease of implementation of the bid price policies make them a popular choice in practice.

One of the traditional approaches for computing bid prices is based on solving a deterministic linear program. This linear program can be viewed as a deterministic approximation to the network revenue management problem that is formulated under the assumption that the numbers of itinerary requests are known in advance and they take on their expected values. In the deterministic linear program, there exists one constraint for each flight leg and the right sides of these constraints are the remaining leg capacities. Therefore, the optimal values of the dual variables associated with these capacity constraints are used as bid prices. Nevertheless, since the deterministic linear program assumes that the numbers of itinerary requests are known in advance, it does not accurately capture the temporal dynamics of the arrivals of the itinerary requests.

In this paper, we propose a new method to compute dynamic bid prices that depend on how much time is left until the time of departure. Our goal is to compute bid prices that capture the temporal dynamics of the arrivals of the itinerary requests somewhat more accurately than the deterministic linear program. The linear program described above can be viewed as a deterministic approximation, whereas our method directly works with the dynamic programming formulation of the network revenue management problem. The main idea behind our method is to relax the capacity constraints in the dynamic programming formulation by associating Lagrange multipliers with them. In this case, the optimality equation decomposes by the time periods and we obtain a simple expression for the value function. We show that a good set of values for the Lagrange multipliers can be obtained by solving a linear program. Since our Lagrange multipliers depend on how much time is left until departure, the bid prices that we obtain also depend on how much time is left until departure.

Our method shares many of the appealing features of the traditional linear programming approach for computing bid prices. To begin with, it is known that the optimal objective value of the deterministic linear program provides an upper bound on the optimal total expected revenue and we show that our method also provides such an upper bound. Furthermore, a popular method to compute dynamic bid prices is based on decomposing the network revenue management problem into a sequence of single leg revenue management problems. This idea is known as dynamic programming decomposition and the bid prices computed by the deterministic linear program serve as a starting point for this method; see Section 3.4.4 in Talluri and van Ryzin (2004). As a matter of fact, it is possible to show that dynamic programming decomposition computes dynamic and capacity-dependent bid prices that depend not only on how much time is left until departure, but also on how many units of capacity are left on the flight legs. We show that the bid prices computed by our method can also be used in a dynamic programming decomposition-like idea to compute dynamic and capacity-dependent bid prices. Finally, Talluri and van Ryzin (1998) show that the bid prices computed by the deterministic linear program are asymptotically optimal as the capacities on the flight legs and the expected numbers of itinerary requests increase linearly with the same rate. It is possible to show that the bid prices computed by our method are also asymptotically optimal in the same sense. We do not pursue this result here, but a related technical report provides the details; see Topaloglu and Kunnumkal (2006). In addition to these parallels between our method and the deterministic linear program, we show that the upper bound on the optimal total expected revenue provided by our method is tighter than the one provided by the linear program. Furthermore, computational experiments indicate that the bid prices computed by our method can perform significantly better than those computed by the deterministic linear program.

Network revenue management is an active area of research. Simpson (1989) and Williamson (1992) were the first to use the deterministic linear program to compute bid prices. Talluri and van Ryzin (1998) give a careful study of bid price policies and show the asymptotic optimality result mentioned above for the bid prices computed by the deterministic linear program. Talluri and van Ryzin (1999) propose a randomized version of the deterministic linear program that uses samples of the numbers of the itinerary requests and this randomization significantly improves the performance. Their goal is to alleviate the shortcoming that the deterministic linear program uses only the expected numbers of the itinerary requests and does not pay attention to the probability distributions. Bertsimas and Popescu (2003) compute bid prices by using the change in the optimal objective value of the deterministic linear program induced by a change in the right sides of certain constraints. By doing so, they try to capture the total opportunity cost of the leg capacities consumed by an itinerary request more accurately. In their paper, they also show how to extend the deterministic linear program to handle cancellations and no shows. Section 3.4.4 in Talluri and van Ryzin (2004) gives a description of the dynamic programming decomposition idea that can be used to decompose the network revenue management problem into a sequence of single leg revenue management problems so as to compute dynamic and capacity-dependent bid prices. As mentioned above, the bid prices computed by the deterministic linear program play an important role in dynamic programming decomposition.

Two papers in the literature are particularly related to our work. First, Adelman (2007) uses linear approximations to the value functions in the dynamic programming formulation of the network revenue management problem. To choose the slope parameters of these approximations, he plugs them into a linear program that represents the dynamic programming formulation of the problem. The number of constraints in this linear program is exponential in the number of flight legs, but the number of decision variables is manageable and he solves this linear program by using column generation on its dual. Similar to our dynamic bid prices, the policy obtained from his linear value function approximations turns out to be a dynamic bid price policy. Second, Topaloglu (2009) computes bid prices by decomposing the network revenue management problem into a sequence of single leg revenue management problems. He uses a Lagrangian relaxation idea, but he does not relax the capacity constraints like we do. Instead, he observes that if an itinerary request is accepted, then the capacities on all of the flight legs that are in this itinerary have to be consumed. He relaxes this requirement so that it becomes allowable to individually accept or reject the capacities on the flight legs that are in the requested itinerary. Under this relaxation, the network revenue management problem decomposes by the flight legs. In our demand model, we assume that each customer arrives into the system with the intention of purchasing one particular itinerary and the decision maker decides whether to accept or reject this itinerary request. This is known as the independent demand model. In reality, there may be many different itineraries that are acceptable to a customer. In this case, the decision maker decides which group of itineraries to make available for purchase. Each customer observes the group of itineraries available for purchase and makes a choice. There has been a lot of recent attention in the literature towards modeling this kind of customer choice behavior. It turns out that many of the models that work for the independent demand setting can be extended to the customer choice setting. Gallego, Iyengar, Phillips and Dubey (2004) and Liu and van Ryzin (2008) extend the deterministic linear program, Zhang and Adelman (2006) extend the linear value function approximations of Adelman (2007) and Kunnumkal and Topaloglu (2007) extend the decomposition strategy of Topaloglu (2009). The method that we develop in this paper can also be extended to the customer choice setting and we pursue this extension in Kunnumkal and Topaloglu (2008b).

Our main methodological machinery is based on relaxing the complicating constraints in a dynamic program by associating Lagrange multipliers with them. The connections between the duality and control theory appear in the earlier literature; see Varaiya (1998) and Bertsekas (2001). Recently, Hawkins (2003) and Adelman and Mersereau (2008) revive these connections by studying dynamic programs that would decompose by the components of the state variable if a few constraints did not play a linking role. They call these dynamic programs as weakly coupled and show that such dynamic programs are particularly suitable for Lagrangian relaxation ideas. Karmarkar (1981), Cheung and Powell (1996), Castanon (1997) and Kunnumkal and Topaloglu (2008*a*) apply Lagrangian relaxation ideas to the dynamic programs arising in multi-location inventory allocation, fleet management, sensor management and multi-echelon inventory distribution settings.

We make the following research contributions in this paper. 1) We propose a new method to compute dynamic bid prices that depend on how much time is left until departure. 2) We show that our method provides an upper bound on the optimal total expected revenue and this upper bound is tighter than the one provided by the deterministic linear program. 3) Our method is based on relaxing the capacity constraints by associating Lagrange multipliers with them. We show that a good set of Lagrange multipliers can be obtained by solving a linear program. 4) We show that the bid prices computed by our method can be used in a dynamic programming decomposition-like idea to compute dynamic and capacity-dependent bid prices. 5) Computational experiments indicate that the bid prices computed by our method perform noticeably better than those computed by many standard benchmark methods.

The rest of the paper is organized as follows. In Section 1, we formulate the network revenue management problem as a dynamic program. In Section 2, we present the basic Lagrangian relaxation idea. In Section 3, we show that a good set of values for the Lagrange multipliers can be obtained by solving a linear program. In Section 4, we show how to use the bid prices computed by our method as a starting point to decompose the network revenue management problem by the flight legs and to compute dynamic and capacity-dependent bid prices. In Section 5, we contrast our method with other approaches for computing bid prices. In Section 6, we present computational experiments.

1 PROBLEM FORMULATION

We have a set of flight legs that can be used to satisfy the itinerary requests that arrive randomly over time. At each time period, an itinerary request arrives and we have to decide whether to accept or reject the itinerary request. An accepted itinerary request generates a revenue and consumes the capacities on the relevant flight legs. A rejected itinerary request simply leaves the system.

The problem takes place over the finite planning horizon $\mathcal{T} = \{1, \ldots, \tau\}$ and all flight legs depart at time period $\tau + 1$. The set of flight legs is \mathcal{L} and the set of itineraries is \mathcal{J} . The capacity on flight leg *i* is c_i . If we accept a request for itinerary *j*, then we generate a revenue of f_j and consume a_{ij} units of capacity on flight leg *i*. We naturally have $a_{ij} = 0$ when itinerary *j* does not use flight leg *i*. The probability that a request for itinerary *j* arrives at time period *t* is p_{jt} . For notational brevity, we assume that $\sum_{j \in \mathcal{J}} p_{jt} = 1$ for all $t \in \mathcal{T}$ so that there is an itinerary request at every time period with probability 1. If there is a strictly positive probability that no itinerary requests arrive at certain time periods, then we can cover this case simply by defining a fictitious itinerary ψ with $f_{\psi} = 0$, $a_{i\psi} = 0$ for all $i \in \mathcal{L}$ and $p_{\psi t} = 1 - \sum_{j \in \mathcal{J}} p_{jt}$ for all $t \in \mathcal{T}$. We assume that the itinerary requests at different time periods are independent.

We let x_{it} be the remaining capacity on flight leg *i* at time period *t* so that $x_t = \{x_{it} : i \in \mathcal{L}\}$ captures the remaining leg capacities at time period *t*. We capture the decisions at time period *t* by $u_t = \{u_{jt} : j \in \mathcal{J}\}$, where u_{jt} takes value 1 if we accept a request for itinerary *j* at time period *t* and u_{jt} takes value 0 if we reject a request for itinerary *j* at time period *t*. In this case, the set of feasible decisions is given by

$$\mathcal{U}(x_t) = \{ u_t \in \{0, 1\}^{|\mathcal{J}|} : a_{ij} \, u_{jt} \le x_{it} \quad \forall \, i \in \mathcal{L}, \ j \in \mathcal{J} \}.$$

These constraints ensure that if we accept a request for itinerary j at time period t, then the capacity consumed by itinerary j on flight leg i does not exceed the remaining capacity on flight leg i.

Using x_t as the state variable and letting e_i be the $|\mathcal{L}|$ -dimensional unit vector with a 1 in the element corresponding to $i \in \mathcal{L}$, the optimal policy can be found by computing the value functions through the optimality equation

$$V_t(x_t) = \max_{u_t \in \mathcal{U}(x_t)} \left\{ \sum_{j \in \mathcal{J}} p_{jt} \left[f_j \, u_{jt} + V_{t+1} (x_t - u_{jt} \sum_{i \in \mathcal{L}} a_{ij} \, e_i) \right] \right\}.$$
(1)

As a function of the state variable x_t , it is easy to see that the optimal decisions at time period t are given by $\hat{u}_t(x_t) = \{\hat{u}_{jt}(x_t) : j \in \mathcal{J}\}$, where

$$\hat{u}_{jt}(x_t) = \begin{cases} 1 & \text{if } f_j + V_{t+1}(x_t - \sum_{i \in \mathcal{L}} a_{ij} e_i) \ge V_{t+1}(x_t) \text{ and } a_{ij} \le x_{it} \text{ for all } i \in \mathcal{L} \\ 0 & \text{otherwise.} \end{cases}$$
(2)

A complicating factor in the optimality equation in (1) is the constraints $a_{ij} u_{jt} \leq x_{it}$ for all $i \in \mathcal{L}$, $j \in \mathcal{J}$ captured by the feasible set $\mathcal{U}(x_t)$. In particular, if these constraints did not exist, then the optimality equation in (1) would decompose by the time periods. This suggests relaxing the constraints $a_{ij} u_{jt} \leq x_{it}$ for all $i \in \mathcal{L}$, $j \in \mathcal{J}$ by associating Lagrange multipliers with them, in which case the optimality equation in (1) has a simple solution. We build on this idea in the next section.

2 LAGRANGIAN RELAXATION STRATEGY

We propose relaxing the constraints $a_{ij} u_{jt} \leq x_{it}$ for all $i \in \mathcal{L}$, $j \in \mathcal{J}$ in the optimality equation in (1) by associating Lagrange multipliers with them. In particular, we associate the nonnegative Lagrange multipliers $\{\alpha_{ijt} : i \in \mathcal{L}, j \in \mathcal{J}\}$ with the constraints $a_{ij} u_{jt} \leq x_{it}$ for all $i \in \mathcal{L}$, $j \in \mathcal{J}$ and solve the optimality equation

$$V_t^{\alpha}(x_t) = \max_{u_t \in \{0,1\}^{|\mathcal{J}|}} \left\{ \sum_{j \in \mathcal{J}} p_{jt} \left\{ \left[f_j - \sum_{i \in \mathcal{L}} a_{ij} \, \alpha_{ijt} \right] u_{jt} + \sum_{i \in \mathcal{L}} \alpha_{ijt} \, x_{it} + V_{t+1}^{\alpha} (x_t - u_{jt} \sum_{i \in \mathcal{L}} a_{ij} \, e_i) \right\} \right\}.$$
 (3)

We note that the Lagrange multipliers above are scaled by $\{p_{jt} : j \in \mathcal{J}\}$ for notational brevity. If we have $p_{jt} = 0$, then the Lagrange multipliers $\{\alpha_{ijt} : i \in \mathcal{L}\}$ are inconsequential and scaling the Lagrange multipliers in this fashion does not create a complication. We use the superscript $\alpha = \{\alpha_{ijt} : i \in \mathcal{L}, j \in \mathcal{J}, t \in \mathcal{T}\}$ in the value functions to emphasize that the solution to the optimality equation in (3) depends on the Lagrange multipliers. Since the Lagrange multiplier α_{ijt} is associated with the capacity availability constraint for flight leg *i* and itinerary *j* at time period *t*, it roughly captures the opportunity cost of a unit of capacity on flight leg *i* given that there is a request for itinerary *j* at time period *t*. As we shortly demonstrate, α_{ijt} is ultimately related to the bid price associated with flight leg *i* at time period *t*.

The next proposition shows that there is a simple solution to the optimality equation in (3). In the next proposition and throughout the rest of the paper, we let

$$r_{it}^{\alpha} = \sum_{j \in \mathcal{J}} p_{jt} \, \alpha_{ijt} + \ldots + \sum_{j \in \mathcal{J}} p_{j\tau} \, \alpha_{ij\tau}$$
$$L_{jt}^{\alpha} = \left[f_j - \sum_{i \in \mathcal{L}} a_{ij} \, \alpha_{ijt} - \sum_{i \in \mathcal{L}} a_{ij} \, r_{i,t+1}^{\alpha} \right]^+,$$

where we use $[\cdot]^+ = \max\{0, \cdot\}$ and follow the convention that $r_{i,\tau+1}^{\alpha} = 0$. We emphasize that both r_{it}^{α} and L_{jt}^{α} are simple functions of the Lagrange multipliers. Furthermore, noting the interpretation of α_{ijt} above, r_{it}^{α} roughly captures the total expected opportunity cost of a unit of capacity on flight leg *i* over time periods $\{t, \ldots, \tau\}$.

Proposition 1 For all $t \in \mathcal{T}$, we have

$$V_t^{\alpha}(x_t) = \sum_{i \in \mathcal{L}} r_{it}^{\alpha} x_{it} + \sum_{j \in \mathcal{J}} p_{jt} L_{jt}^{\alpha} + \ldots + \sum_{j \in \mathcal{J}} p_{j\tau} L_{j\tau}^{\alpha}.$$
 (4)

Proof We show the result by induction over the time periods. It is easy to show the result for the last time period. Assuming that the result holds for time period t + 1, (3) implies that

$$V_t^{\alpha}(x_t) = \max_{u_t \in \{0,1\}^{|\mathcal{J}|}} \left\{ \sum_{j \in \mathcal{J}} p_{jt} \left\{ \left[f_j - \sum_{i \in \mathcal{L}} a_{ij} \alpha_{ijt} \right] u_{jt} + \sum_{i \in \mathcal{L}} \alpha_{ijt} x_{it} + \sum_{i \in \mathcal{L}} r_{i,t+1}^{\alpha} \left[x_{it} - a_{ij} u_{jt} \right] \right\} \right\} + \sum_{j \in \mathcal{J}} p_{j,t+1} L_{j,t+1}^{\alpha} + \dots + \sum_{j \in \mathcal{J}} p_{j\tau} L_{j\tau}^{\alpha}.$$

Arranging the terms and letting $\mathbf{1}(\cdot)$ be the indicator function, the optimal values of the decision variables $\{u_{jt} : j \in \mathcal{J}\}$ in the problem above are $\{\mathbf{1}(f_j - \sum_{i \in \mathcal{L}} a_{ij} \alpha_{ijt} - \sum_{i \in \mathcal{L}} a_{ij} r_{i,t+1}^{\alpha} \ge 0) : j \in \mathcal{J}\}$ and the result follows from the definition of L_{jt}^{α} and the fact that $r_{it}^{\alpha} = \sum_{j \in \mathcal{J}} p_{jt} \alpha_{ijt} + r_{i,t+1}^{\alpha}$.

Therefore, the value functions computed through the optimality equation in (3) are linear functions of the remaining leg capacities. This is one of the connections between our method and the method proposed by Adelman (2007), as both methods end up using linear value function approximations. In Section 5, we show a result that indicates that the method proposed by Adelman (2007) is potentially stronger in terms of the tightness of the upper bounds. Nevertheless, our computational experiments in Section 6 demonstrate that the upper bounds obtained by the two methods are identical for all of our test problems. Kunnumkal and Topaloglu (2008*b*) extend Proposition 1 to the customer choice setting, but the computation of L_{it}^{α} is significantly more complicated in their case.

Assuming that the value functions $\{V_t^{\alpha}(\cdot) : t \in \mathcal{T}\}$ computed through the optimality equation in (3) are good approximations to the value functions $\{V_t(\cdot) : t \in \mathcal{T}\}$ computed through the optimality equation in (1), we propose making the itinerary acceptance decisions by replacing $\{V_t(\cdot) : t \in \mathcal{T}\}$ on the right side of (2) with $\{V_t^{\alpha}(\cdot) : t \in \mathcal{T}\}$. We shortly dwell on the questions of what we mean by good approximations and how we can choose the Lagrange multipliers so that $\{V_t^{\alpha}(\cdot) : t \in \mathcal{T}\}$ are good approximations to $\{V_t(\cdot) : t \in \mathcal{T}\}$. Leaving these questions aside for the time being, the idea of approximating $\{V_t(\cdot) : t \in \mathcal{T}\}$ in (2) by $\{V_t^{\alpha}(\cdot) : t \in \mathcal{T}\}$ implies that if we have

$$f_j + V_{t+1}^{\alpha}(x_t - \sum_{i \in \mathcal{L}} a_{ij} e_i) \ge V_{t+1}^{\alpha}(x_t)$$

and $a_{ij} \leq x_{it}$ for all $i \in \mathcal{L}$, then we accept a request for itinerary j at time period t. Otherwise, we reject the itinerary request. Since we have $V_{t+1}^{\alpha}(x_t) - V_{t+1}^{\alpha}(x_t - \sum_{i \in \mathcal{L}} a_{ij} e_i) = \sum_{i \in \mathcal{L}} a_{ij} r_{i,t+1}^{\alpha}$ by (4), this idea is equivalent to accepting a request for itinerary j at time period t when we have

$$f_j \ge \sum_{i \in \mathcal{L}} a_{ij} r_{i,t+1}^{\alpha} \tag{5}$$

and $a_{ij} \leq x_{it}$ for all $i \in \mathcal{L}$. Therefore, we can view $r_{i,t+1}^{\alpha}$ as the bid price associated with flight leg i at time period t. Similar to the idea of bid prices described in the introduction, if there is enough capacity and the revenue from an itinerary request exceeds the sum of the bid prices associated with the flight legs that are in the requested itinerary, then we accept the itinerary request. It is important to note that the bid price associated with a flight leg in (5) depends on how much time is left until departure. In particular, since the Lagrange multipliers are nonnegative and $r_{it}^{\alpha} = \sum_{j \in \mathcal{J}} p_{jt} \alpha_{ijt} + r_{i,t+1}^{\alpha}$, we have $r_{it}^{\alpha} \geq r_{i,t+1}^{\alpha} \geq \ldots \geq r_{i,\tau+1}^{\alpha}$ and the bid price associated with flight leg i decreases as we approach the departure time of the flight legs. This is in agreement with the intuitive expectation that the opportunities to utilize the leg capacities remain.

3 Choosing the Lagrange Multipliers

In this section, we consider the question of how to choose the Lagrange multipliers. We begin with the next proposition, which shows that we can obtain upper bounds on the value functions by solving the optimality equation in (3).

Proposition 2 If the Lagrange multipliers are nonnegative, then we have $V_t(x_t) \leq V_t^{\alpha}(x_t)$ for all $t \in \mathcal{T}$.

Proof We show the result by induction over the time periods. It is easy to show the result for the last time period. Assuming that the result holds for time period t + 1 and letting $\hat{u}_t = \{\hat{u}_{jt} : j \in \mathcal{J}\}$ be an optimal solution to problem (1), we have

$$\begin{aligned} V_t(x_t) &= \sum_{j \in \mathcal{J}} p_{jt} \Big[f_j \, \hat{u}_{jt} + V_{t+1}(x_t - \hat{u}_{jt} \sum_{i \in \mathcal{L}} a_{ij} \, e_i) \Big] \\ &\leq \sum_{j \in \mathcal{J}} p_{jt} \Big\{ \Big[f_j - \sum_{i \in \mathcal{L}} a_{ij} \, \alpha_{ijt} \Big] \hat{u}_{jt} + \sum_{i \in \mathcal{L}} \alpha_{ijt} \, x_{it} + V_{t+1}^{\alpha}(x_t - \hat{u}_{jt} \sum_{i \in \mathcal{L}} a_{ij} \, e_i) \Big\} \leq V_t^{\alpha}(x_t), \end{aligned}$$

where the first inequality follows from the induction assumption and the fact that $\alpha_{ijt} \ge 0$ and $a_{ij} \hat{u}_{jt} \le x_{it}$ for all $i \in \mathcal{L}, j \in \mathcal{J}$ and the second inequality follows from the fact that \hat{u}_t is a feasible but not necessarily an optimal solution to problem (3).

Proposition 2 is a standard upper bound result that often arises in the applications of Lagrangian relaxation ideas. Cheung and Powell (1996) and Kunnumkal and Topaloglu (2008*b*) show similar results in fleet management and network revenue management with customer choice settings. Adelman and Mersereau (2008) show a similar result for weakly coupled dynamic programs over an infinite horizon, but their proof simplifies significantly when one considers the finite horizon case.

Since the initial leg capacities are given by $c = \{c_i : i \in \mathcal{L}\}$, the optimal total expected revenue over the planning horizon is $V_1(c)$. Proposition 2 implies that $V_1(c)$ is bounded from above by $V_1^{\alpha}(c)$ as long as the Lagrange multipliers are nonnegative. Therefore, to obtain the tightest possible upper bound on $V_1(c)$, we can solve the problem

$$\min_{\alpha \ge 0} \left\{ V_1^{\alpha}(c) \right\}. \tag{6}$$

It turns out that we can obtain an optimal solution to problem (6) by solving a linear program. To see this, we first note that

$$V_1^{\alpha}(c) = \sum_{i \in \mathcal{L}} r_{i1}^{\alpha} c_i + \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} p_{jt} L_{jt}^{\alpha}$$

$$\tag{7}$$

by Proposition 1. In this case, the next proposition shows that the linear program

$$\min \quad \sum_{i \in \mathcal{L}} c_i \,\rho_{i1} + \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} p_{jt} \,\Gamma_{jt} \tag{8}$$

subject to
$$\Gamma_{jt} \ge f_j - \sum_{i \in \mathcal{L}} a_{ij} \, \alpha_{ijt} - \sum_{i \in \mathcal{L}} a_{ij} \, \rho_{i,t+1} \qquad \forall j \in \mathcal{J}, \, t \in \mathcal{T} \setminus \{\tau\}$$

$$\tag{9}$$

$$\Gamma_{j\tau} \ge f_j - \sum_{i \in \mathcal{L}} a_{ij} \,\alpha_{ij\tau} \qquad \qquad \forall j \in \mathcal{J}$$
⁽¹⁰⁾

$$\rho_{it} = \sum_{j \in \mathcal{J}} p_{jt} \, \alpha_{ijt} + \ldots + \sum_{j \in \mathcal{J}} p_{j\tau} \, \alpha_{ij\tau} \qquad \forall i \in \mathcal{L}, \ t \in \mathcal{T}$$
(11)

$$\rho_{it} \text{ is free}, \Gamma_{jt} \ge 0, \alpha_{ijt} \ge 0 \qquad \qquad \forall i \in \mathcal{L}, \ j \in \mathcal{J}, \ t \in \mathcal{T}$$
(12)

is equivalent to problem (6). In the next proposition and throughout the rest of the paper, we use $\hat{\zeta}$ to denote the optimal objective value of problem (8)-(12).

Proposition 3 We have $\hat{\zeta} = \min_{\alpha \ge 0} \{V_1^{\alpha}(c)\}.$

Proof If $\hat{\alpha} = \{\hat{\alpha}_{ijt} : i \in \mathcal{L}, j \in \mathcal{J}, t \in \mathcal{T}\}$ is an optimal solution to problem (6), then the definitions of r_{it}^{α} and L_{jt}^{α} imply that $\{r_{it}^{\hat{\alpha}} : i \in \mathcal{L}, t \in \mathcal{T}\}, \{L_{jt}^{\hat{\alpha}} : j \in \mathcal{J}, t \in \mathcal{T}\}, \{\hat{\alpha}_{ijt} : i \in \mathcal{L}, j \in \mathcal{J}, t \in \mathcal{T}\}$ is a feasible solution to problem (8)-(12). In this case, (7) implies that we have $\hat{\zeta} \leq \sum_{i \in \mathcal{L}} c_i r_{i1}^{\hat{\alpha}} + \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} p_{jt} L_{jt}^{\hat{\alpha}} = V_1^{\hat{\alpha}}(c) = \min_{\alpha \geq 0} \{V_1^{\alpha}(c)\}.$

On the other hand, if $\{\hat{\rho}_{it}: i \in \mathcal{L}, t \in \mathcal{T}\}, \{\hat{\Gamma}_{jt}: j \in \mathcal{J}, t \in \mathcal{T}\}, \hat{\alpha} = \{\hat{\alpha}_{ijt}: i \in \mathcal{L}, j \in \mathcal{J}, t \in \mathcal{T}\}$ is an optimal solution to problem (8)-(12), then we have $\hat{\rho}_{it} = r_{it}^{\hat{\alpha}}$ for all $i \in \mathcal{L}, t \in \mathcal{T}$ by constraints (11) and the definition of r_{it}^{α} . Constraints (9) and the fact that problem (8)-(12) is a minimization problem imply that $\hat{\Gamma}_{jt} = [f_j - \sum_{i \in \mathcal{L}} a_{ij} \hat{\alpha}_{ijt} - \sum_{i \in \mathcal{L}} a_{ij} \hat{\rho}_{i,t+1}]^+ = [f_j - \sum_{i \in \mathcal{L}} a_{ij} \hat{\alpha}_{ijt} - \sum_{i \in \mathcal{L}} a_{ij} r_{i,t+1}^{\hat{\alpha}}]^+ = L_{jt}^{\hat{\alpha}}$ for all $j \in$ $\mathcal{J}, t \in \mathcal{T} \setminus \{\tau\}$, where the last equality follows from the definition of L_{jt}^{α} . Similarly, constraints (10) and the fact that problem (8)-(12) is a minimization problem imply that $\hat{\Gamma}_{j\tau} = [f_j - \sum_{i \in \mathcal{L}} a_{ij} \hat{\alpha}_{ij\tau}]^+ = L_{j\tau}^{\hat{\alpha}}$ for all $j \in \mathcal{J}$. In this case, we have $\hat{\zeta} = \sum_{i \in \mathcal{L}} c_i \hat{\rho}_{i1} + \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} p_{jt} \hat{\Gamma}_{jt} = \sum_{i \in \mathcal{L}} r_{i1}^{\hat{\alpha}} c_i + \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} p_{jt} L_{jt}^{\hat{\alpha}} =$ $V_1^{\hat{\alpha}}(c) \geq \min_{\alpha \geq 0} \{V_1^{\alpha}(c)\}.$

Therefore, the optimal objective value of problem (6) can be obtained by solving problem (8)-(12). The proof of Proposition 3 also implies that if $\{\hat{\rho}_{it} : i \in \mathcal{L}, t \in \mathcal{T}\}, \{\hat{\Gamma}_{jt} : j \in \mathcal{J}, t \in \mathcal{T}\}, \hat{\alpha} = \{\hat{\alpha}_{ijt} : i \in \mathcal{L}, j \in \mathcal{J}, t \in \mathcal{T}\}$ is an optimal solution to problem (8)-(12), then $\hat{\alpha}$ is an optimal solution to problem (6).

The fact that $V_1^{\alpha}(c)$ is linear in c plays a crucial role in the tractability of the linear program in (8)-(12). In particular, it is important to note that the numbers of decision variables and constraints in problem (8)-(12) do not depend on the capacities of the flight legs. On the other hand, Adelman and Mersereau (2008) show that if $V_1^{\alpha}(c)$ were not linear in c, then the problem of finding the tightest possible upper bound could still be formulated as a linear program, but the numbers of decision variables and constraints in this linear program would increase linearly with the capacities. Problem (PL) in Adelman and Mersereau (2008) shows what this linear program would look like for a weakly coupled dynamic program over an infinite horizon and one can extend problem (PL) to the finite horizon case.

To gain some insight into the structure of problem (8)-(12), we associate the dual variables $\{y_{jt} : j \in \mathcal{J}, t \in \mathcal{T}\}$ with constraints (9)-(10) and the dual variables $\{z_{it} : i \in \mathcal{L}, t \in \mathcal{T}\}$ with constraints (11). In this case, the dual of problem (8)-(12) can be written as

$$\begin{split} \hat{\zeta} &= \max \quad \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} f_j \, y_{jt} \\ \text{subject to} \quad a_{ij} \, y_{jt} - p_{jt} \, z_{i1} - \ldots - p_{jt} \, z_{it} \leq 0 \qquad \forall i \in \mathcal{L}, \ j \in \mathcal{J}, \ t \in \mathcal{T} \\ z_{i1} &= c_i \qquad \forall i \in \mathcal{L} \\ \sum_{j \in \mathcal{J}} a_{ij} \, y_{j,t-1} + z_{it} &= 0 \qquad \forall i \in \mathcal{L}, \ t \in \mathcal{T} \setminus \{1\} \\ y_{jt} \leq p_{jt} \qquad \forall j \in \mathcal{J}, \ t \in \mathcal{T} \\ y_{jt} \geq 0, z_{it} \text{ is free} \qquad \forall i \in \mathcal{L}, \ j \in \mathcal{J}, \ t \in \mathcal{T}. \end{split}$$

Substituting for the decision variables $\{z_{it} : i \in \mathcal{L}, t \in \mathcal{T}\}$ in the first set of constraints above by using the second and third sets of constraints, the problem above becomes

$$\hat{\zeta} = \max \quad \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} f_j \, y_{jt} \tag{13}$$

subject to
$$\sum_{k \in \mathcal{J}} p_{jt} a_{ik} y_{k1} + \ldots + \sum_{k \in \mathcal{J}} p_{jt} a_{ik} y_{k,t-1} + a_{ij} y_{jt} \le p_{jt} c_i \quad \forall i \in \mathcal{L}, \ j \in \mathcal{J}, \ t \in \mathcal{T}$$
(14)

$$y_{jt} \le p_{jt} \qquad \qquad \forall j \in \mathcal{J}, \ t \in \mathcal{T}$$
(15)

$$y_{jt} \ge 0 \qquad \qquad \forall j \in \mathcal{J}, \ t \in \mathcal{T}.$$
 (16)

We have $V_1(c) \leq \min_{\alpha \geq 0} \{V_1^{\alpha}(c)\}$ by Proposition 2 and $\hat{\zeta} = \min_{\alpha \geq 0} \{V_1^{\alpha}(c)\}$ by Proposition 3. Therefore, we can obtain an upper bound on the optimal total expected revenue by solving problem (13)-(16).

There is an appealing interpretation for problem (13)-(16). The decision variable y_{jt} in this problem corresponds to the probability that we accept a request for itinerary j at time period t. In this case, the objective function accounts for the total expected revenue over the planning horizon. Constraints (15) ensure that the probability that we accept a request for itinerary j at time period t does not exceed the probability that a request for itinerary j arrives at time period t. The interpretation of constraints (14) is a bit more intricate. If we write constraints (14) as

$$a_{ij} \frac{y_{jt}}{p_{jt}} \le c_i - \sum_{k \in \mathcal{J}} a_{ik} y_{k1} - \ldots - \sum_{k \in \mathcal{J}} a_{ik} y_{k,t-1} \qquad \forall i \in \mathcal{L}, \ j \in \mathcal{J}, \ t \in \mathcal{T},$$
(17)

then the right side of constraints (17) is the expected remaining capacity on flight leg i at time period t. The term y_{jt}/p_{jt} on the left side of constraints (17) is the conditional probability that we accept a request for itinerary j at time period t given that there is a request for itinerary j at time period t. Therefore, constraints (17) ensure that the expected capacity consumed on flight leg i, given that there is a request for itinerary j at time period t, does not exceed the expected remaining capacity on flight leg i at time period t.

Kunnumkal and Topaloglu (2008b) extend problem (13)-(16) to the customer choice setting. In this case, the number of constraints remains essentially the same, but the number of decision variables increases exponentially with the number of itineraries. They deal with the large number of decision variables by using column generation, but their column generation subproblem ends up being a nontrivial mixed integer linear program.

4 CAPACITY-DEPENDENT BID PRICES

The method that we describe in Sections 2 and 3 computes dynamic bid prices. However, one intuitively expects that the opportunity cost of a unit of capacity on a flight leg should not only decrease as we approach the departure time, but it should also increase as the capacity on the flight leg becomes scarce. In other words, the bid prices should depend both on how much time is left until departure and on how many units of capacity are left on the flight legs. In this section, we develop a method that computes dynamic and capacity-dependent bid prices by decomposing the network revenue management problem

into a number of single leg revenue management problems. This method is closely related to the popular dynamic programming decomposition idea; see Section 3.4.4 in Talluri and van Ryzin (2004).

We let $\hat{\alpha} = {\hat{\alpha}_{ijt} : i \in \mathcal{L}, j \in \mathcal{J}, t \in \mathcal{T}}$ be the optimal values of the dual variables associated with constraints (14) in problem (13)-(16). We choose a flight leg *i* and letting $\bar{\mathcal{L}}_i = \mathcal{L} \setminus {i}$ for notational brevity, we relax constraints (14) for all other flight legs by associating the dual multipliers ${\hat{\alpha}_{ljt} : l \in \bar{\mathcal{L}}_i, j \in \mathcal{J}, t \in \mathcal{T}}$ with them. In this case, the objective function of problem (13)-(16) can be written as

$$\sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{J}}f_j y_{jt} + \sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{J}}\sum_{l\in\bar{\mathcal{L}}_i} \left[p_{jt} c_l - \sum_{k\in\mathcal{J}}p_{jt} a_{lk} y_{k1} - \dots - \sum_{k\in\mathcal{J}}p_{jt} a_{lk} y_{k,t-1} - a_{lj} y_{jt}\right]\hat{\alpha}_{ljt}$$

Arranging the terms, it is easy to see that the expression above becomes

$$\sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{J}} \left[f_j - \sum_{l\in\bar{\mathcal{L}}_i} a_{lj} \,\hat{\alpha}_{ljt} \right] y_{jt} + \sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{J}}\sum_{l\in\bar{\mathcal{L}}_i} p_{jt} \,c_l \,\hat{\alpha}_{ljt} \\ - \sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{J}}\sum_{l\in\bar{\mathcal{L}}_i} a_{lk} \,y_{kt} \Big[\sum_{j\in\mathcal{J}} p_{jt} \,\hat{\alpha}_{lj,t+1} + \ldots + \sum_{j\in\mathcal{J}} p_{jt} \,\hat{\alpha}_{lj\tau} \Big].$$

Noting the definition of r_{it}^{α} and arranging the terms above once more, the objective function of problem (13)-(16) can finally be written as

$$\sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{J}}\left[f_j-\sum_{l\in\bar{\mathcal{L}}_i}a_{lj}\,\hat{\alpha}_{ljt}-\sum_{l\in\bar{\mathcal{L}}_i}a_{lj}\,r_{l,t+1}^{\hat{\alpha}}\right]y_{jt}+\sum_{l\in\bar{\mathcal{L}}_i}r_{l1}^{\hat{\alpha}}\,c_l.$$

Therefore, the duality theory implies that the linear program

$$\hat{\zeta} = \max \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} \left[f_j - \sum_{l \in \bar{\mathcal{L}}_i} a_{lj} \,\hat{\alpha}_{ljt} - \sum_{l \in \bar{\mathcal{L}}_i} a_{lj} \, r_{l,t+1}^{\hat{\alpha}} \right] y_{jt} + \sum_{l \in \bar{\mathcal{L}}_i} r_{l1}^{\hat{\alpha}} \, c_l \tag{18}$$

subject to
$$\sum_{k \in \mathcal{J}} p_{jt} a_{ik} y_{k1} + \ldots + \sum_{k \in \mathcal{J}} p_{jt} a_{ik} y_{k,t-1} + a_{ij} y_{jt} \le p_{jt} c_i \qquad \forall j \in \mathcal{J}, \ t \in \mathcal{T}$$
(19)

$$y_{jt} \le p_{jt} \qquad \qquad \forall j \in \mathcal{J}, \ t \in \mathcal{I}$$
 (20)

$$y_{jt} \ge 0 \qquad \qquad \forall j \in \mathcal{J}, \ t \in \mathcal{T} \qquad (21)$$

has the same optimal objective value as problem (13)-(16).

Ignoring the constant term $\sum_{l \in \bar{\mathcal{L}}_i} r_{l1}^{\hat{\alpha}} c_l$ in the objective function, problem (18)-(21) has the same structure as problem (13)-(16), but problem (18)-(21) focuses on a single leg revenue management problem that takes place over flight leg *i* under the assumption that

$$f_j - \sum_{l \in \bar{\mathcal{L}}_i} a_{lj} \, \hat{\alpha}_{ljt} - \sum_{l \in \bar{\mathcal{L}}_i} a_{lj} \, r_{l,t+1}^{\hat{\alpha}}$$

is the revenue associated with itinerary j at time period t. Therefore, $\hat{\zeta} - \sum_{l \in \bar{L}_i} r_{l1}^{\hat{\alpha}} c_l$ is an upper bound on the optimal total expected revenue for the single leg revenue management problem that takes place over flight leg i. On the other hand, we can obtain the optimal total expected revenue for the single leg revenue management problem that takes place over flight leg i by solving the optimality equation

$$v_{it}(x_{it}) = \max_{u_t \in \mathcal{U}_i(x_{it})} \left\{ \sum_{j \in \mathcal{J}} p_{jt} \left\{ \left[f_j - \sum_{l \in \bar{\mathcal{L}}_i} a_{lj} \,\hat{\alpha}_{ljt} - \sum_{l \in \bar{\mathcal{L}}_i} a_{lj} \,r_{l,t+1}^{\hat{\alpha}} \right] u_{jt} + v_{i,t+1}(x_{it} - a_{ij} \,u_{jt}) \right\} \right\},$$
(22)

where we let $\mathcal{U}_i(x_{it}) = \{u_t \in \{0,1\}^{|\mathcal{J}|} : a_{ij} u_{jt} \leq x_{it} \forall j \in \mathcal{J}\}$ and use an optimality equation that is similar to the one in (1), but focus only on flight leg *i*.

We have $v_{i1}(c_i) \leq \hat{\zeta} - \sum_{l \in \bar{\mathcal{L}}_i} r_{l1}^{\hat{\alpha}} c_l$ by the discussion in the paragraph above. Furthermore, the next proposition shows that $V_1(c) \leq v_{i1}(c_i) + \sum_{l \in \bar{\mathcal{L}}_i} r_{l1}^{\hat{\alpha}} c_l$. Therefore, we have

$$V_1(c) \le v_{i1}(c_i) + \sum_{l \in \hat{\mathcal{L}}_i} r_{l1}^{\hat{\alpha}} c_l \le \hat{\zeta}$$
(23)

so that $v_{i1}(c_i) + \sum_{l \in \bar{\mathcal{L}}_i} r_{l1}^{\hat{\alpha}} c_l$ is an upper bound on the optimal total expected revenue and this upper bound is tighter than the one provided by the optimal objective value of problem (13)-(16).

Proposition 4 If $\hat{\alpha} = {\hat{\alpha}_{ijt} : i \in \mathcal{L}, j \in \mathcal{J}, t \in \mathcal{T}}$ are the optimal values of the dual variables associated with constraints (14) in problem (13)-(16), then we have $V_t(x_t) \leq v_{it}(x_{it}) + \sum_{l \in \bar{\mathcal{L}}_i} r_{lt}^{\hat{\alpha}} x_{lt}$ for all $t \in \mathcal{T}$.

Proof We show the result by induction over the time periods. It is easy to show the result for the last time period. Assuming that the result holds for time period t + 1 and letting $\hat{u}_t = \{\hat{u}_{jt} : j \in \mathcal{J}\}$ be an optimal solution to problem (1), we have

$$\begin{split} V_{t}(x_{t}) &= \sum_{j \in \mathcal{J}} p_{jt} \Big[f_{j} \, \hat{u}_{jt} + V_{t+1}(x_{t} - \hat{u}_{jt} \sum_{i \in \mathcal{L}} a_{ij} \, e_{i}) \Big] \\ &\leq \sum_{j \in \mathcal{J}} p_{jt} \Big\{ \Big[f_{j} - \sum_{l \in \bar{\mathcal{L}}_{i}} a_{lj} \, \hat{\alpha}_{ljt} \Big] \hat{u}_{jt} + \sum_{l \in \bar{\mathcal{L}}_{i}} \hat{\alpha}_{ljt} \, x_{lt} + v_{i,t+1}(x_{it} - a_{ij} \, \hat{u}_{jt}) + \sum_{l \in \bar{\mathcal{L}}_{i}} r_{l,t+1}^{\hat{\alpha}} [x_{lt} - a_{lj} \, \hat{u}_{jt}] \Big\} \\ &= \sum_{j \in \mathcal{J}} p_{jt} \Big\{ \Big[f_{j} - \sum_{l \in \bar{\mathcal{L}}_{i}} a_{lj} \, \hat{\alpha}_{ljt} - \sum_{l \in \bar{\mathcal{L}}_{i}} a_{lj} \, r_{l,t+1}^{\hat{\alpha}} \Big] \hat{u}_{jt} + v_{i,t+1}(x_{it} - a_{ij} \, \hat{u}_{jt}) \Big\} + \sum_{l \in \bar{\mathcal{L}}_{i}} r_{lt}^{\hat{\alpha}} \, x_{lt}, \\ &\leq v_{it}(x_{it}) + \sum_{l \in \bar{\mathcal{L}}_{i}} r_{lt}^{\hat{\alpha}} \, x_{lt}, \end{split}$$

where the first inequality follows from the induction assumption and the fact that $\hat{\alpha}_{ljt} \geq 0$ and $a_{lj} \hat{u}_{jt} \leq x_{lt}$ for all $l \in \bar{\mathcal{L}}_i$, $j \in \mathcal{J}$, the second equality follows from the fact that $r_{it}^{\alpha} = \sum_{j \in \mathcal{J}} p_{jt} \alpha_{ijt} + r_{i,t+1}^{\alpha}$ and the second inequality follows from the fact that \hat{u}_t is a feasible but not necessarily an optimal solution to problem (22).

Dynamic programming decomposition ideas date back to Belobaba (1987), but the fact that these ideas can provide upper bounds on the optimal total expected revenue is recently shown by Zhang and Adelman (2006). Proposition 4 shows that a dynamic programming decomposition idea in conjunction with the dynamic bid prices computed by our method can provide upper bounds on the optimal total expected revenue. Our induction proof mimics the one in Zhang and Adelman (2006), but the relaxation argument that we use on problem (13)-(16) is new and it clearly demonstrates why we need to associate the revenue $f_j - \sum_{l \in \bar{\mathcal{L}}_i} a_{lj} \hat{\alpha}_{ljt} - \sum_{l \in \bar{\mathcal{L}}_i} a_{lj} r_{l,t+1}^{\hat{\alpha}}$ with itinerary j at time period t. The term $f_j - \sum_{l \in \bar{\mathcal{L}}_i} a_{lj} \hat{\alpha}_{ljt} - \sum_{l \in \bar{\mathcal{L}}_i} a_{lj} r_{l,t+1}^{\hat{\alpha}}$ is quite nonintuitive and it is not easy to come up with this term without following our relaxation argument. Using Proposition 4 for all of the flight legs, the tightest possible upper bound on the optimal total expected revenue is $\min_{i \in \mathcal{L}} \{v_{i1}(c_i) + \sum_{l \in \bar{\mathcal{L}}_i} r_{l1}^{\hat{\alpha}} c_l\}$. Furthermore, we can collect the one-dimensional value functions $\{v_{it}(\cdot) : i \in \mathcal{L}, t \in \mathcal{T}\}$ together to construct the separable value function approximation $\tilde{V}_t(x_t) = \sum_{i \in \mathcal{L}} v_{it}(x_{it})$ for all $t \in \mathcal{T}$. In this case, we make the itinerary acceptance decisions by replacing $\{V_t(\cdot) : t \in \mathcal{T}\}$ on the right side of (2) with $\{\sum_{i \in \mathcal{L}} v_{it}(\cdot) : t \in \mathcal{T}\}$. Therefore, if we have

$$f_j + \sum_{i \in \mathcal{L}} v_{i,t+1}(x_{it} - a_{ij}) \ge \sum_{i \in \mathcal{L}} v_{i,t+1}(x_{it})$$

and $a_{ij} \leq x_{it}$ for all $i \in \mathcal{L}$, then we accept a request for itinerary j at time period t. This is essentially the same approximation used by the dynamic programming decomposition idea. Noting that $v_{i,t+1}(x_{it}) - v_{i,t+1}(x_{it} - a_{ij}) = \sum_{q=1}^{a_{ij}} [v_{i,t+1}(x_{it} + 1 - q) - v_{i,t+1}(x_{it} - q)]$, the decision rule above is equivalent to accepting a request for itinerary j at time period t when we have

$$f_j \ge \sum_{i \in \mathcal{L}} \sum_{q=1}^{a_{ij}} \left[v_{i,t+1}(x_{it}+1-q) - v_{i,t+1}(x_{it}-q) \right]$$
(24)

and $a_{ij} \leq x_{it}$ for all $i \in \mathcal{L}$.

We can view $v_{i,t+1}(x_{it}) - v_{i,t+1}(x_{it}-1)$ as the bid price associated with the x_{it} -th unit of capacity on flight leg *i* at time period *t*. Comparing the decision rule in (24) with the one in (5), we note that the bid prices in (5) are dynamic, whereas the bid prices in (24) are both dynamic and capacity-dependent. It is also possible to show that $\{v_{it}(\cdot) : i \in \mathcal{L}, t \in \mathcal{T}\}$ are concave functions and this implies that the bid price associated with a flight leg increases as the capacity on the flight leg becomes scarce. This is in agreement with the intuitive expectation that we mention at the beginning of this section.

5 OTHER APPROACHES FOR COMPUTING BID PRICES

There exist a number of other approaches in the network revenue management literature that can be used to compute bid prices. In this section, we review some of these approaches and compare them with the two methods that we develop in Sections 2-4. All of the approaches that we review in this section are used as benchmark methods in our computational experiments.

5.1 DETERMINISTIC LINEAR PROGRAM

A traditional approach for computing bid prices involves solving a deterministic linear program. Letting w_j be the number of requests for itinerary j that we plan to accept over the planning horizon, this linear program has the form

$$\max \quad \sum_{j \in \mathcal{J}} f_j \, w_j \tag{25}$$

subject to
$$\sum_{i \in \mathcal{I}} a_{ij} w_j \le c_i \qquad \forall i \in \mathcal{L}$$
 (26)

$$w_j \le \sum_{t \in \mathcal{T}} p_{jt} \qquad \forall j \in \mathcal{J}$$
 (27)

$$w_j \ge 0 \qquad \forall j \in \mathcal{J}.$$
 (28)

Constraints (26) ensure that the numbers of itinerary requests that we plan to accept do not violate the leg capacities, whereas constraints (27) ensure that we do not plan to accept more itinerary requests than the expected numbers of itinerary requests. The linear program above dates back to Simpson (1989) and Williamson (1992).

Letting $\{\hat{\mu}_i : i \in \mathcal{L}\}$ be the optimal values of the dual variables associated with constraints (26) in problem (25)-(28), we can use $\hat{\mu}_i$ as an estimate of the opportunity cost of a unit of capacity on flight leg *i*. In other words, we can use $\hat{\mu}_i$ as the bid price associated with flight leg *i*. In this case, we accept a request for itinerary *j* at time period *t* when we have

$$f_j \ge \sum_{i \in \mathcal{L}} a_{ij} \,\hat{\mu}_i \tag{29}$$

and $a_{ij} \leq x_{it}$ for all $i \in \mathcal{L}$. Letting $\tilde{V}_t(x_t) = \sum_{i \in \mathcal{L}} \hat{\mu}_i x_{it}$ for all $t \in \mathcal{T}$, since $\tilde{V}_{t+1}(x_t) - \tilde{V}_{t+1}(x_t - \sum_{i \in \mathcal{L}} a_{ij} e_i) = \sum_{i \in \mathcal{L}} a_{ij} \hat{\mu}_i$, it is easy to see that the decision rule in (29) is equivalent to approximating $\{V_t(\cdot) : t \in \mathcal{T}\}$ in (2) by $\{\tilde{V}_t(\cdot) : t \in \mathcal{T}\}$. We emphasize that the bid prices in (29) do not depend on how much time is left until departure or on how many units of capacity are left on the flight legs.

It is possible to show that the optimal objective value of problem (25)-(28) provides an upper bound on the optimal total expected revenue; see Bertsimas and Popescu (2003). In other words, letting \hat{Z} be the optimal objective value of problem (25)-(28), we have $V_1(c) \leq \hat{Z}$. The next proposition shows that $V_1(c) \leq \hat{\zeta} \leq \hat{Z}$. Therefore, the upper bound on the optimal total expected revenue provided by problem (13)-(16) is tighter than the one provided by problem (25)-(28).

Proposition 5 We have $V_1(c) \leq \hat{\zeta} \leq \hat{Z}$.

Proof The proof follows from an aggregation argument similar to the proof of Theorem 1 in Adelman (2007). Propositions 2 and 3 show that $V_1(c) \leq \hat{\zeta}$ and we focus only on the second inequality. We let $\hat{y} = \{\hat{y}_{jt} : j \in \mathcal{J}, t \in \mathcal{T}\}$ be an optimal solution to problem (13)-(16) and define the solution $\hat{w} = \{\hat{w}_j : j \in \mathcal{J}\}$ as $\hat{w}_j = \sum_{t \in \mathcal{T}} \hat{y}_{jt}$ for all $j \in \mathcal{J}$. Since \hat{y} is a feasible solution to problem (13)-(16) and $\sum_{j \in \mathcal{J}} p_{j\tau} = 1$, adding constraints (14) for flight leg *i* and time period τ over all $j \in \mathcal{J}$, we have

$$c_i = \sum_{j \in \mathcal{J}} p_{j\tau} c_i \ge \sum_{j \in \mathcal{J}} p_{j\tau} \Big[\sum_{k \in \mathcal{J}} a_{ik} \, \hat{y}_{k1} + \ldots + \sum_{k \in \mathcal{J}} a_{ik} \, \hat{y}_{k,\tau-1} \Big] + \sum_{j \in \mathcal{J}} a_{ij} \, \hat{y}_{j\tau} = \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} a_{ij} \, \hat{y}_{jt}$$

for all $i \in \mathcal{L}$. Similarly, adding constraints (15) for itinerary j over all $t \in \mathcal{T}$, we have $\sum_{t \in \mathcal{T}} \hat{y}_{jt} \leq \sum_{t \in \mathcal{T}} p_{jt}$ for all $j \in \mathcal{J}$. Therefore, \hat{w} is a feasible solution to problem (25)-(28) and we have $\hat{\zeta} = \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} f_j \hat{y}_{jt} = \sum_{j \in \mathcal{J}} f_j \hat{w}_j \leq \hat{Z}$.

Comparing problem (25)-(28) with problem (13)-(16), we note that the main difference between the two problems is in the way in which they capture the capacity availabilities. Problem (13)-(16) has one capacity constraint for each flight leg, for each itinerary and for each time period, whereas problem (25)-(28) has one capacity constraint for each flight leg. Therefore, the capacity constraints in problem (13)-(16) operate at a more disaggregate level than those in problem (25)-(28) and the upper bound provided by problem (13)-(16) ends up being tighter.

5.2 Dynamic Programming Decomposition

This method is similar to the one in Section 4 and it computes dynamic and capacity-dependent bid prices by decomposing the network revenue management problem into a number of single leg revenue management problems. In particular, letting $\{\hat{\mu}_i : i \in \mathcal{L}\}$ be the optimal values of the dual variables associated with constraints (26) in problem (25)-(28), we consider the single leg revenue management problem that takes place over flight leg *i* under the assumption that $f_j - \sum_{l \in \bar{\mathcal{L}}_i} a_{lj} \hat{\mu}_l$ is the revenue associated with itinerary *j*. The optimal total expected revenue for the single leg revenue management problem that takes place over flight leg *i* can be obtained by solving the optimality equation

$$\vartheta_{it}(x_{it}) = \max_{u_t \in \mathcal{U}_i(x_{it})} \left\{ \sum_{j \in \mathcal{J}} p_{jt} \left\{ \left[f_j - \sum_{l \in \bar{\mathcal{L}}_i} a_{lj} \,\hat{\mu}_l \right] u_{jt} + \vartheta_{i,t+1}(x_{it} - a_{ij} \, u_{jt}) \right\} \right\}.$$
(30)

Using a relaxation argument that is similar to the one in Section 4, but working with problem (25)-(28) and dual multipliers $\{\hat{\mu}_l : l \in \bar{\mathcal{L}}_i\}$ instead of problem (13)-(16) and dual multipliers $\{\hat{\alpha}_{ljt} : l \in \bar{\mathcal{L}}_i, j \in \mathcal{J}, t \in \mathcal{T}\}$, it is possible to show that

$$V_1(c) \le \vartheta_{i1}(c_i) + \sum_{l \in \bar{\mathcal{L}}_i} \hat{\mu}_l \, c_l \le \hat{Z}.$$
(31)

Therefore, we can solve the optimality equation in (30) to obtain an upper bound on the optimal total expected revenue that is tighter than the one provided by problem (25)-(28). We note that the chain of inequalities in (31) is analogous to the one in (23). This result was first shown by Zhang and Adelman (2006), but without using the relaxation argument that we use in this paper.

Repeating this approach for all of the flight legs, the tightest possible upper bound on the optimal total expected revenue is $\min_{i \in \mathcal{L}} \{\vartheta_{i1}(c_i) + \sum_{l \in \bar{\mathcal{L}}_i} \hat{\mu}_l c_l\}$. Furthermore, we can collect the one-dimensional value functions $\{\vartheta_{it}(\cdot) : i \in \mathcal{L}, t \in \mathcal{T}\}$ together to construct the separable value function approximation $\tilde{V}_t(x_t) = \sum_{i \in \mathcal{L}} \vartheta_{it}(x_{it})$ for all $t \in \mathcal{T}$; see Section 3.4.4 in Talluri and van Ryzin (2004). In this case, replacing $\{V_t(\cdot) : t \in \mathcal{T}\}$ on the right side of (2) with $\{\sum_{i \in \mathcal{L}} \vartheta_{it}(\cdot) : t \in \mathcal{T}\}$, if we have

$$f_j + \sum_{i \in \mathcal{L}} \vartheta_{i,t+1}(x_{it} - a_{ij}) \ge \sum_{i \in \mathcal{L}} \vartheta_{i,t+1}(x_{it})$$

and $a_{ij} \leq x_{it}$ for all $i \in \mathcal{L}$, then we accept a request for itinerary j at time period t. We note that the decision rule above is equivalent to using dynamic and capacity-dependent bid prices.

5.3 Decomposition by Revenue Allocation

This approach was developed by Topaloglu (2009) to compute dynamic and capacity-dependent bid prices. Kunnumkal and Topaloglu (2007) show that it is possible to extend this approach to the customer choice setting. The main idea behind this approach is to allocate the revenue associated with an itinerary among the different flight legs. In particular, we let β_{ijt} be the portion of the revenue generated from accepting a request for itinerary j at time period t that is allocated to flight leg i. We do not specify yet how the revenue allocations are chosen, but they satisfy

$$\sum_{i \in \mathcal{L}} \beta_{ijt} = f_j \qquad \forall j \in \mathcal{J}, \ t \in \mathcal{T}.$$
(32)

Allocating the revenues in this fashion immediately allows us to formulate single leg revenue management problems. In the single leg revenue management problem that takes place over flight leg *i*, the revenue associated with itinerary *j* at time period *t* is β_{ijt} . Therefore, we can solve the optimality equation

$$\nu_{it}^{\beta}(x_{it}) = \max_{u_t \in \mathcal{U}_i(x_{it})} \left\{ \sum_{j \in \mathcal{J}} p_{jt} \Big[\beta_{ijt} \, u_{jt} + \nu_{i,t+1}^{\beta}(x_{it} - a_{ij} \, u_{jt}) \Big] \right\}$$
(33)

to obtain the optimal total expected revenue for the single leg revenue management that takes place over flight leg *i*. We use the superscript $\beta = \{\beta_{ijt} : i \in \mathcal{L}, j \in \mathcal{J}, t \in \mathcal{T}\}$ in the value functions to emphasize that the solution to the optimality equation above depends on the revenue allocations.

The next proposition shows that we can obtain upper bounds on the value functions by solving the optimality equation in (33). Proposition 1 in Kunnumkal and Topaloglu (2007) shows this result in the customer choice setting. Since the proof of this result is crucial for demonstrating why the revenue allocations need to satisfy (32), we simplify the proof in Kunnumkal and Topaloglu (2007) to capture the independent demand setting here.

Proposition 6 If β satisfies (32), then we have $V_t(x_t) \leq \sum_{i \in \mathcal{L}} \nu_{it}^{\beta}(x_{it})$ for all $t \in \mathcal{T}$.

Proof We show the result by induction over the time periods. It is easy to show the result for the last time period. Assuming that the result holds for time period t + 1 and letting $\hat{u}_t = \{\hat{u}_{jt} : j \in \mathcal{J}\}$ be an optimal solution to problem (1), we have

$$\begin{aligned} V_t(x_t) &= \sum_{j \in \mathcal{J}} p_{jt} \Big[\sum_{i \in \mathcal{L}} \beta_{ijt} \, \hat{u}_{jt} + V_{t+1}(x_t - \hat{u}_{jt} \sum_{i \in \mathcal{L}} a_{ij} \, e_i) \Big] \\ &\leq \sum_{j \in \mathcal{J}} p_{jt} \Big[\sum_{i \in \mathcal{L}} \beta_{ijt} \, \hat{u}_{jt} + \sum_{i \in \mathcal{L}} \nu_{i,t+1}^\beta(x_{it} - a_{ij} \, \hat{u}_{jt}) \Big] \leq \sum_{i \in \mathcal{L}} \nu_{it}^\beta(x_{it}), \end{aligned}$$

where the first equality follows from (32), the first inequality follows from the induction assumption and the second inequality follows from the fact that \hat{u}_t is a feasible but not necessarily an optimal solution to problem (33).

Therefore, $V_1(c)$ is bounded from above by $\sum_{i \in \mathcal{L}} \nu_{i1}^{\beta}(c_i)$ as long as the revenue allocations satisfy (32). To obtain the tightest possible upper bound on $V_1(c)$, we can solve the problem

$$\min_{\beta \in \mathcal{F}} \left\{ \sum_{i \in \mathcal{L}} \nu_{i1}^{\beta}(c_i) \right\},\tag{34}$$

where $\mathcal{F} = \{\beta : \sum_{i \in \mathcal{L}} \beta_{ijt} = f_j \ \forall j \in \mathcal{J}, t \in \mathcal{T}\}$. Proposition 3 in Topaloglu (2009) shows that $\nu_{i1}^{\beta}(c_i)$ is a convex function of β and we can solve problem (34) by using subgradient optimization or Benders decomposition; see Wolsey (1998) and Ruszczynski (2003).

We note that problem (34) is analogous to problem (6). However, there does not exist a closed form expression for $\sum_{i \in \mathcal{L}} \nu_{i1}^{\beta}(c_i)$ comparable to (4) and solving problem (34) is computationally much more expensive than solving problem (6). The next proposition shows that the extra computational burden pays off and the upper bound on the optimal total expected revenue provided by problem (34) is tighter than the one provided by problem (6). **Proposition 7** If $\hat{\alpha} = {\hat{\alpha}_{ijt} : i \in \mathcal{L}, j \in \mathcal{J}, t \in \mathcal{T}}$ are the optimal values of the dual variables associated with constraints (14) in problem (13)-(16), then we have

$$\min_{\beta \in \mathcal{F}} \left\{ \sum_{i \in \mathcal{L}} \nu_{i1}^{\beta}(c_i) \right\} \leq \min_{i \in \mathcal{L}} \left\{ v_{i1}(c_i) + \sum_{l \in \bar{\mathcal{L}}_i} r_{l1}^{\hat{\alpha}} c_l \right\} \leq \min_{\alpha \geq 0} \{ V_1^{\alpha}(c) \}.$$

Proof Proposition 3 and the second inequality in (23) show that the second inequality above holds and we focus only on the first inequality here. We let $\hat{\alpha}$ be an optimal solution to problem (6). We choose a flight leg *i* and let $\hat{\beta}_{ijt} = f_j - \sum_{l \in \bar{\mathcal{L}}_i} a_{lj} \hat{\alpha}_{ljt} - \sum_{l \in \bar{\mathcal{L}}_i} a_{lj} r_{l,t+1}^{\hat{\alpha}}$ for all $j \in \mathcal{J}, t \in \mathcal{T}$ and $\hat{\beta}_{ljt} = a_{lj} \hat{\alpha}_{ljt} + a_{lj} r_{l,t+1}^{\hat{\alpha}}$ for all $l \in \bar{\mathcal{L}}_i, j \in \mathcal{J}, t \in \mathcal{T}$. We note that $\hat{\beta} = \{\hat{\beta}_{ijt} : i \in \mathcal{L}, j \in \mathcal{J}, t \in \mathcal{T}\} \in \mathcal{F}$. The proof shows that $\sum_{l \in \mathcal{L}} \nu_{lt}^{\hat{\beta}}(x_{lt}) \leq v_{it}(x_{it}) + \sum_{l \in \bar{\mathcal{L}}_i} r_{lt}^{\hat{\alpha}} x_{lt}$ for all $t \in \mathcal{T}$. In this case, the result follows from the fact that flight leg *i* is arbitrary and $\hat{\beta} \in \mathcal{F}$.

First, if we let $\beta_{ijt} = \hat{\beta}_{ijt}$ for all $j \in \mathcal{J}$, $t \in \mathcal{T}$ in (33), then the optimality equations in (22) and (33) become identical. Therefore, we have $\nu_{it}^{\hat{\beta}}(x_{it}) = v_{it}(x_{it})$ for all $t \in \mathcal{T}$. Second, we show by induction over the time periods that $\nu_{lt}^{\hat{\beta}}(x_{lt}) \leq r_{lt}^{\hat{\alpha}} x_{lt}$ for all $l \in \bar{\mathcal{L}}_i$, $t \in \mathcal{T}$. It is easy to show the result for the last time period. Assuming that the result holds for time period t + 1, we let $\hat{u}_t = \{\hat{u}_{jt} : j \in \mathcal{J}\}$ be an optimal solution to problem (33) when we solve this problem for flight leg l with $\beta = \hat{\beta}$. We have

$$\begin{split} \nu_{lt}^{\hat{\beta}}(x_{lt}) &= \sum_{j \in \mathcal{J}} p_{jt} \Big[[a_{lj} \,\hat{\alpha}_{ljt} + a_{lj} \, r_{l,t+1}^{\hat{\alpha}}] \,\hat{u}_{jt} + \nu_{l,t+1}^{\hat{\beta}}(x_{lt} - a_{lj} \,\hat{u}_{jt}) \Big] \\ &\leq \sum_{j \in \mathcal{J}} p_{jt} \Big[[a_{lj} \,\hat{\alpha}_{ljt} + a_{lj} \, r_{l,t+1}^{\hat{\alpha}}] \,\hat{u}_{jt} + r_{l,t+1}^{\hat{\alpha}} \, [x_{lt} - a_{lj} \, \hat{u}_{jt}] \Big] \\ &\leq \sum_{j \in \mathcal{J}} p_{jt} \,\hat{\alpha}_{ljt} \, x_{lt} + r_{l,t+1}^{\hat{\alpha}} \, x_{lt} = r_{lt}^{\hat{\alpha}} \, x_{lt}, \end{split}$$

where the first inequality follows from the induction assumption, the second inequality follows from the fact that $\hat{\alpha}_{ljt} \geq 0$ and $a_{lj} \hat{u}_{jt} \leq x_{lt}$ for all $j \in \mathcal{J}$ and the second equality follows from the fact that $r_{it}^{\alpha} = \sum_{j \in \mathcal{J}} p_{jt} \alpha_{ijt} + r_{i,t+1}^{\alpha}$.

Kunnumkal and Topaloglu (2007) use the argument above to show a similar result in the customer choice setting. If their result were translated to the independent demand setting, then it would read as $\min_{\beta \in \mathcal{F}} \{\sum_{i \in \mathcal{L}} \nu_{i1}^{\beta}(c_i)\} \leq \min_{i \in \mathcal{L}} \{\vartheta_{i1}(c_i) + \sum_{l \in \bar{\mathcal{L}}_i} \hat{\mu}_l c_l\} \leq \hat{Z}, \text{ where } \{\vartheta_{it}(\cdot) : i \in \mathcal{L}, t \in \mathcal{T}\} \text{ are computed}$ through the optimality equation in (30), $\{\hat{\mu}_i : i \in \mathcal{L}\}$ are the optimal values of the dual variables associated with constraints (26) in problem (25)-(28) and \hat{Z} is the optimal objective value of problem (25)-(28). Therefore, we can interpret Proposition 7 as the analogue of their result for the case where we compute dynamic bid prices through problem (13)-(16).

Letting $\hat{\beta}$ be an optimal solution to problem (34), we make the itinerary acceptance decisions by replacing $\{V_t(\cdot): t \in \mathcal{T}\}$ on the right side of (2) with $\{\sum_{i \in \mathcal{L}} \nu_{it}^{\hat{\beta}}(\cdot): t \in \mathcal{T}\}$. Therefore, if we have

$$f_j + \sum_{i \in \mathcal{L}} \nu_{i,t+1}^{\hat{\beta}}(x_{it} - a_{ij}) \ge \sum_{i \in \mathcal{L}} \nu_{i,t+1}^{\hat{\beta}}(x_{it})$$

$$(35)$$

and $a_{ij} \leq x_{it}$ for all $i \in \mathcal{L}$, then we accept a request for itinerary j at time period t. Similar to the decision rules in Sections 4 and 5.2, the decision rule above is equivalent to using dynamic and capacity-dependent bid prices.

5.4 LINEAR VALUE FUNCTION APPROXIMATIONS

Similar to the method that we describe in Sections 2 and 3, this method is directed at computing dynamic bid prices and it was proposed by Adelman (2007). Zhang and Adelman (2006) extend this method to the customer choice setting. We begin by letting $C = \max_{i \in \mathcal{L}} \{c_i\}$ and $\mathcal{C} = \{0, \ldots, C\}$ so that the remaining capacity on each flight leg is always in the set \mathcal{C} and we can use $\mathcal{C}^{|\mathcal{L}|}$ as the state space in the optimality equation in (1). In this case, Adelman (2007) shows that $V_1(c)$ can be computed by solving the linear program

min $V_1(c)$

sι

abject to
$$V_t(x_t) \ge \sum_{j \in \mathcal{J}} p_{jt} \left[f_j \, u_{jt} + V_{t+1}(x_t - u_{jt} \sum_{i \in \mathcal{L}} a_{ij} \, e_i) \right] \quad \forall \, x_t \in \mathcal{C}^{|\mathcal{L}|}, \, \, u_t \in \mathcal{U}(x_t), \, t \in \mathcal{T} \setminus \{\tau\}$$

 $V_{\tau}(x_{\tau}) \ge \sum_{j \in \mathcal{J}} p_{j\tau} \, f_j \, u_{j\tau} \qquad \qquad \forall \, x_{\tau} \in \mathcal{C}^{|\mathcal{L}|}, \, \, u_{\tau} \in \mathcal{U}(x_{\tau}),$

where $\{V_t(x_t) : x_t \in \mathcal{C}^{|\mathcal{L}|}, t \in \mathcal{T}\}$ are the decision variables. One approach to deal with the large number of decision variables in the problem above is to approximate the value functions by linear functions of the form $\hat{V}_t(x_t) = \theta_t + \sum_{i \in \mathcal{L}} \gamma_{it} x_{it}$. To decide what values to choose for $\{\theta_t : t \in \mathcal{T}\}$ and $\{\gamma_{it} : i \in \mathcal{L}, t \in \mathcal{T}\}$, we replace $V_t(x_t)$ in the problem above with $\theta_t + \sum_{i \in \mathcal{L}} \gamma_{it} x_{it}$ to obtain the linear program

$$\min \quad \theta_{1} + \sum_{i \in \mathcal{L}} \gamma_{i1} c_{i}$$
(36)
subject to
$$\theta_{t} + \sum_{i \in \mathcal{L}} \gamma_{it} x_{it} \geq \sum_{j \in \mathcal{J}} p_{jt} \left[f_{j} u_{jt} + \theta_{t+1} + \sum_{i \in \mathcal{L}} \gamma_{i,t+1} \left[x_{it} - a_{ij} u_{jt} \right] \right]$$
$$\forall x_{t} \in \mathcal{C}^{|\mathcal{L}|}, \ u_{t} \in \mathcal{U}(x_{t}), \ t \in \mathcal{T} \setminus \{\tau\}$$
(37)
$$\theta_{\tau} + \sum_{i \in \mathcal{L}} \gamma_{i\tau} x_{i\tau} \geq \sum_{j \in \mathcal{J}} p_{j\tau} f_{j} u_{j\tau}$$
$$\forall x_{\tau} \in \mathcal{C}^{|\mathcal{L}|}, \ u_{\tau} \in \mathcal{U}(x_{\tau}),$$
(38)

where $\{\theta_t : t \in \mathcal{T}\}\$ and $\{\gamma_{it} : i \in \mathcal{L}, t \in \mathcal{T}\}\$ are the decision variables. The number of decision variables in problem (36)-(38) is manageable and we can deal with the large number of constraints by using column generation on the dual.

Letting $\{\hat{\theta}_t : t \in \mathcal{T}\}$ and $\{\hat{\gamma}_{it} : i \in \mathcal{L}, t \in \mathcal{T}\}$ be an optimal solution to problem (36)-(38) and $\hat{V}_t(x_t) = \hat{\theta}_t + \sum_{i \in \mathcal{L}} \hat{\gamma}_{it} x_{it}$, we approximate the value functions $\{V_t(\cdot) : t \in \mathcal{T}\}$ in (2) by $\{\hat{V}_t(\cdot) : t \in \mathcal{T}\}$. In this case, noting that $\hat{V}_{t+1}(x_t) - \hat{V}_{t+1}(x_t - \sum_{i \in \mathcal{L}} a_{ij} e_i) = \sum_{i \in \mathcal{L}} a_{ij} \hat{\gamma}_{i,t+1}$, we accept a request for itinerary j at time period t when we have

$$f_j \ge \sum_{i \in \mathcal{L}} a_{ij} \,\hat{\gamma}_{i,t+1} \tag{39}$$

and $a_{ij} \leq x_{it}$ for all $i \in \mathcal{L}$. The decision rule above is similar to the one in (5) and it is equivalent to using dynamic bid prices. Adelman (2007) also shows that $V_1(c) \leq \hat{\theta}_1 + \sum_{i \in \mathcal{L}} \hat{\gamma}_{i1} c_i$ so that we can obtain an upper bound on the optimal total expected revenue by solving problem (36)-(38). The next proposition shows that this upper bound is tighter than the one provided by the optimal objective value of problem (13)-(16). **Proposition 8** If $\{\hat{\theta}_t : t \in \mathcal{T}\}$ and $\{\hat{\gamma}_{it} : i \in \mathcal{L}, t \in \mathcal{T}\}$ are the optimal values of the decision variables in problem (36)-(38), then we have $\hat{\theta}_1 + \sum_{i \in \mathcal{L}} \hat{\gamma}_{i1} c_i \leq \hat{\zeta}$.

Proof We let $\hat{\alpha}$ be an optimal solution to problem (6). Since we have $V_1^{\hat{\alpha}}(c) = \hat{\zeta}$ by Proposition 3, we equivalently show that $\hat{\theta}_1 + \sum_{i \in \mathcal{L}} \hat{\gamma}_{i1} c_i \leq V_1^{\hat{\alpha}}(c)$. We let $\tilde{\theta}_t = \sum_{j \in \mathcal{J}} p_{jt} L_{jt}^{\hat{\alpha}} + \ldots + \sum_{j \in \mathcal{J}} p_{j\tau} L_{j\tau}^{\hat{\alpha}}$ and $\tilde{\gamma}_{it} = r_{it}^{\hat{\alpha}}$ for all $i \in \mathcal{L}, t \in \mathcal{T}$. By Proposition 1, we have $\tilde{\theta}_1 + \sum_{i \in \mathcal{L}} \tilde{\gamma}_{i1} c_i = \sum_{j \in \mathcal{J}} p_{j1} L_{j1}^{\hat{\alpha}} + \ldots + \sum_{j \in \mathcal{J}} p_{j\tau} L_{j\tau}^{\hat{\alpha}} + \sum_{i \in \mathcal{L}} r_{i1}^{\hat{\alpha}} c_i = V_1^{\hat{\alpha}}(c)$. Therefore, if we can show that the solution $\{\tilde{\theta}_t : t \in \mathcal{T}\}$ and $\{\tilde{\gamma}_{it} : i \in \mathcal{L}, t \in \mathcal{T}\}$ is feasible to problem (36)-(38), then we have a feasible solution to problem (36)-(38) that yields the objective value $V_1^{\hat{\alpha}}(c)$ and the result follows.

For all $x_t \in \mathcal{C}^{|\mathcal{L}|}, u_t \in \mathcal{U}(x_t), t \in \mathcal{T} \setminus \{\tau\}$, we have

$$\begin{split} \tilde{\theta}_t + \sum_{i \in \mathcal{L}} \tilde{\gamma}_{it} \, x_{it} &= \sum_{j \in \mathcal{J}} p_{jt} \left[f_j - \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\alpha}_{ijt} - \sum_{i \in \mathcal{L}} a_{ij} \, r_{i,t+1}^{\hat{\alpha}} \right]^+ + \tilde{\theta}_{t+1} + \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} p_{jt} \, \hat{\alpha}_{ijt} \, x_{it} + \sum_{i \in \mathcal{L}} r_{i,t+1}^{\hat{\alpha}} \, x_{it} \\ &\geq \sum_{j \in \mathcal{J}} p_{jt} \left[f_j - \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\alpha}_{ijt} - \sum_{i \in \mathcal{L}} a_{ij} \, r_{i,t+1}^{\hat{\alpha}} \right] u_{jt} + \tilde{\theta}_{t+1} + \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} p_{jt} \, \hat{\alpha}_{ijt} \, x_{it} + \sum_{i \in \mathcal{L}} r_{i,t+1}^{\hat{\alpha}} \, x_{it} \\ &\geq \sum_{j \in \mathcal{J}} p_{jt} \left[f_j \, u_{jt} + \tilde{\theta}_{t+1} + \sum_{i \in \mathcal{L}} r_{i,t+1}^{\hat{\alpha}} \left[x_{it} - a_{ij} \, u_{jt} \right] \right], \end{split}$$

where the equality follows from the definition of L_{jt}^{α} and the fact that $r_{it}^{\alpha} = \sum_{j \in \mathcal{J}} p_{jt} \alpha_{ijt} + r_{i,t+1}^{\alpha}$, the first inequality follows from the fact that $u_{jt} \in \{0,1\}$ for all $j \in \mathcal{J}$ and the second inequality follows from the fact that $a_{ij} u_{jt} \leq x_{it}$ and $\hat{\alpha}_{ijt} \geq 0$ for all $i \in \mathcal{L}, j \in \mathcal{J}$. The last expression in the chain of inequalities above is equal to $\sum_{j \in \mathcal{J}} p_{jt} [f_j u_{jt} + \tilde{\theta}_{t+1} + \sum_{i \in \mathcal{L}} \tilde{\gamma}_{i,t+1} [x_{it} - a_{ij} u_{jt}]]$, which implies that the solution $\{\tilde{\theta}_t : t \in \mathcal{T}\}$ and $\{\tilde{\gamma}_{it} : i \in \mathcal{L}, t \in \mathcal{T}\}$ satisfies constraints (37). A similar argument shows that this solution also satisfies constraints (38) and the result follows. \Box

5.5 CAPACITY-DEPENDENT BID PRICES THROUGH LINEAR VALUE FUNCTION APPROXIMATIONS

Similar to the methods in Sections 4, 5.2 and 5.3, this method computes dynamic and capacity-dependent bid prices, but it builds on the optimal solution to problem (36)-(38). In particular, letting $\{\hat{\theta}_t : t \in \mathcal{T}\}$ and $\{\hat{\gamma}_{it} : i \in \mathcal{L}, t \in \mathcal{T}\}$ be an optimal solution to problem (36)-(38), Zhang and Adelman (2006) propose solving the optimality equation

$$\begin{aligned} v_{it}(x_{it}) &= \max \quad \sum_{j \in \mathcal{J}} p_{jt} \Big\{ \Big[f_j - \sum_{l \in \bar{\mathcal{L}}_i} a_{lj} \, \hat{\gamma}_{l,t+1} \Big] \, u_{jt} + v_{i,t+1} (x_{it} - a_{ij} \, u_{jt}) \Big\} + \sum_{l \in \bar{\mathcal{L}}_i} \left[\hat{\gamma}_{lt} - \hat{\gamma}_{l,t+1} \right] x_{lt} \\ \text{subject to} \quad a_{lj} \, u_{jt} \leq x_{lt} \qquad \forall l \in \mathcal{L} \ , j \in \mathcal{J} \\ u_{jt} \in \{0, 1\} \qquad \forall j \in \mathcal{J} \\ x_{lt} \in \mathcal{C} \qquad \forall l \in \bar{\mathcal{L}}_i, \end{aligned}$$

where we follow the convention that $\hat{\gamma}_{i,\tau+1} = 0$. In the problem above, the capacity on flight leg *i* is the argument of the value function and it is assumed to be fixed, but the capacities on the other flight legs are decision variables. Therefore, this problem ends up being a nontrivial integer program. Zhang and Adelman (2006) show that $V_1(c) \leq v_{i1}(c_i) + \sum_{l \in \bar{\mathcal{L}}_i} \hat{\gamma}_{i1} c_i$ so that we can obtain an upper bound on the optimal total expected revenue by solving the optimality equation above. Furthermore, collecting the one-dimensional value functions $\{v_{it}(\cdot) : i \in \mathcal{L}, t \in \mathcal{T}\}$ together, we can construct the separable value function approximation $\tilde{V}_t(x_t) = \sum_{i \in \mathcal{L}} v_{it}(x_{it})$ for all $t \in \mathcal{T}$. In this case, we make the itinerary acceptance decisions by replacing $\{V_t(\cdot) : t \in \mathcal{T}\}$ on the right side of (2) with $\{\sum_{i \in \mathcal{L}} v_{it}(\cdot) : t \in \mathcal{T}\}$. Similar to the methods in Sections 4, 5.2 and 5.3, it is easy to see that this idea is equivalent to using dynamic and capacity-dependent bid prices.

6 Computational Experiments

In this section, we compare the performances of the bid prices computed by numerous benchmark methods. Our goal is to demonstrate the benefits from using more sophisticated methods that compute dynamic and capacity-dependent bid prices.

6.1 Benchmark Methods

We compare the performances of the bid prices computed by the following eight benchmark methods.

Lagrangian Relaxation with Dynamic Bid Prices (LRD) This is the solution method that we develop in Sections 2 and 3, but our practical implementation recomputes the bid prices n times over the planning horizon by solving problem (6) at time periods $\{1 + k \tau/n : k = 0, 1, ..., n-1\}$. In particular, given the remaining leg capacities at time period $1 + k \tau/n$, we solve the problem $\min_{\alpha \ge 0} \{V_{1+k\tau/n}^{\alpha}(x_{1+k\tau/n})\}$ to obtain an optimal solution $\hat{\alpha}^{1+k\tau/n}$. We replace α in the decision rule in (5) with $\hat{\alpha}^{1+k\tau/n}$ and follow this decision rule until we recompute the bid prices. In all of our computational experiments, we use n = 5 or n = 50.

Lagrangian Relaxation with Dynamic and Capacity-Dependent Bid Prices (LRDC) This is the solution method that we develop in Section 4, but similar to LRD, our practical implementation recomputes the bid prices n times over the planning horizon. In particular, given the remaining leg capacities at time period $1 + k \tau/n$, we solve the problem $\min_{\alpha \geq 0} \{V_{1+k\tau/n}^{\alpha}(x_{1+k\tau/n})\}$ to obtain an optimal solution $\hat{\alpha}^{1+k\tau/n}$. We replace $\hat{\alpha}$ in the optimality equation in (22) with $\hat{\alpha}^{1+k\tau/n}$ and solve this optimality equation to compute the value functions $\{v_{it}(\cdot) : i \in \mathcal{L}, t \in \mathcal{T}\}$. We follow the decision rule in (24) until we recompute the bid prices.

Deterministic Linear Program (DLP) This is the solution method that we describe in Section 5.1, but our practical implementation also recomputes the bid prices n times over the planning horizon. In particular, given the remaining leg capacities at time period $1 + k \tau/n$, we replace the right side of constraints (26) with $\{x_{i,1+k\tau/n} : i \in \mathcal{L}\}$ and the right side of constraints (27) with $\{\sum_{t=1+k\tau/n}^{\tau} p_{jt} : j \in \mathcal{J}\}$ and solve problem (25)-(28). Letting $\{\hat{\mu}_i^{1+k\tau/n} : i \in \mathcal{L}\}$ be the optimal values of the dual variables associated with constraints (26), we replace $\{\hat{\mu}_i : i \in \mathcal{L}\}$ in the decision rule in (29) with $\{\hat{\mu}_i^{1+k\tau/n} : i \in \mathcal{L}\}$ and follow this decision rule until we recompute the bid prices.

Randomized Linear Program (RLP) This solution method was proposed by Talluri and van Ryzin (1999). Noting that DLP uses only the expected numbers of itinerary requests, RLP tries to make up for this deficiency by working with actual samples. In particular, we let D_{jt} be the number of requests

for itinerary j at time period t so that we have $\mathbb{P}\{D_{jt}=1\} = p_{jt}$ and $\mathbb{P}\{D_{jt}=0\} = 1-p_{jt}$. To compute the bid prices at time period $1+k\tau/n$, we generate S independent samples of $D = \{D_{jt}: j \in \mathcal{J}, t \in \mathcal{T}\}$, which we denote by $\tilde{D}^s = \{\tilde{D}_{jt}^s: j \in \mathcal{J}, t \in \mathcal{T}\}$ for $s = 1, \ldots, S$. Given the remaining leg capacities at time period $1+k\tau/n$, we replace the right side of constraints (26) with $\{x_{i,1+k\tau/n}: i \in \mathcal{L}\}$ and the right side of constraints (27) with $\{\sum_{t=1+k\tau/n}^{\tau} \tilde{D}_{jt}^s: j \in \mathcal{J}\}$ and solve problem (25)-(28). Letting $\hat{Z}^{1+k\tau/n}(\tilde{D}^s)$ be the optimal objective value of this problem and $\{\hat{\mu}_i^{1+k\tau/n}(\tilde{D}^s): i \in \mathcal{L}\}$ be the optimal values of the dual variables associated with constraints (26), we use $\sum_{s=1}^{S} \hat{\mu}_i^{1+k\tau/n}(\tilde{D}^s)/S$ as the bid price associated with flight leg i until we recompute the bid prices.

It is also possible to show that $V_1(c) \leq \mathbb{E}\{\hat{Z}^1(D)\}$. Therefore, RLP provides an upper bound on the optimal total expected revenue, but computing $\mathbb{E}\{\hat{Z}^1(D)\}$ requires estimating the expectation through simulation.

Dynamic Programming Decomposition (DPD) This is the solution method that we describe in Section 5.2, but our practical implementation recomputes the bid prices n times over the planning horizon by using an approach similar to the one used by LRDC.

Decomposition by Revenue Allocation (DRA) This is the solution method that we describe in Section 5.3, but similar to LRDC and DPD, our practical implementation recomputes the bid prices n times over the planning horizon. In particular, given the remaining leg capacities at time period $1 + k \tau/n$, we solve the problem $\min_{\beta \in \mathcal{F}} \{\sum_{i \in \mathcal{L}} \nu_{i,1+k\tau/n}^{\beta}(x_{i,1+k\tau/n})\}$ to obtain an optimal solution $\hat{\beta}^{1+k\tau/n}$. We replace $\hat{\beta}$ in the decision rule in (35) with $\hat{\beta}^{1+k\tau/n}$ and follow this decision rule until we recompute the bid prices.

Linear Approximations with Dynamic Bid Prices (LAD) This is the solution method that we describe in Section 5.4, but similar to LRD and DLP, our practical implementation recomputes the bid prices ntimes over the planning horizon. In particular, given the remaining leg capacities at time period $1+k \tau/n$, we replace the objective function of problem (36)-(38) with $\theta_{1+k\tau/n} + \sum_{i \in \mathcal{L}} \gamma_{i,1+k\tau/n} x_{i,1+k\tau/n}$ and solve this problem to obtain an optimal solution $\{\hat{\theta}_t^{1+k\tau/n}: t \in \mathcal{T}\}$ and $\{\hat{\gamma}_{it}^{1+k\tau/n}: i \in \mathcal{L}, t \in \mathcal{T}\}$. We replace $\{\hat{\gamma}_{it}: i \in \mathcal{L}, t \in \mathcal{T}\}$ in the decision rule in (39) with $\{\hat{\gamma}_{it}^{1+k\tau/n}: i \in \mathcal{L}, t \in \mathcal{T}\}$ and follow this decision rule until we recompute the bid prices.

Linear Approximations with Dynamic and Capacity-Dependent Bid Prices (LADC) This is the solution method that we describe in Section 5.5, but our practical implementation recomputes the bid prices n times over the planning horizon by using an approach similar to the one used by LRDC.

6.2 EXPERIMENTAL SETUP

We consider two types of airline networks in our computational experiments. The first airline network includes one hub and N spokes. There are two flight legs associated with each spoke. One of these flight legs is from the spoke to the hub and the other one is from the hub to the spoke. There is a high-fare and a low-fare itinerary that connect each possible origin-destination pair. Therefore, the first airline network includes 2N flight legs and 2N(N+1) itineraries, and an itinerary includes at most two flight legs. The structure of the first airline network for N = 6 is shown on the left side of Figure 1. The second airline network includes two hubs and N spokes. The first half of the spokes are connected to the first hub and the second half of the spokes are connected to the second hub. There are two flight legs associated with each spoke. One of these flight legs is from the spoke to the hub and the other one is from the hub to the spoke. There are also two flight legs that connect the hubs in each direction. We sample 50 to 150 origin-destination pairs among the set of all possible origin-destination pairs and assume that there is a high-fare and a low-fare itinerary that connect each sampled origin-destination pair. Therefore, the second airline network includes 2N + 2 flights and 100 to 300 itineraries, and an itinerary includes at most three flight legs. The structure of the second airline network for N = 10 is shown on the right side of Figure 1.

For both airline networks, the revenue associated with a high-fare itinerary is κ times larger than the revenue associated with the corresponding low-fare itinerary. We generate the arrival probabilities $\{p_{jt} : j \in \mathcal{J}, t \in \mathcal{T}\}$ in such a manner that the probability of having a request for a high-fare itinerary increases over time, whereas the probability of having a request for a low-fare itinerary decreases over time. To do this, we randomly generate one probability for each origin-destination pair in the airline network such that the sum of these probabilities over all origin-destination pairs is equal to 1. If we let P_{od} be the probability that we generate for origin-destination pair (o, d) and $\lambda(t)$ be an increasing function of t taking values over [0, 1], then the probability that there is a request at time period t for the high-fare itinerary associated with origin-destination pair (o, d) is $\lambda(t) P_{od}$ and the probability that there is a request at time period t for the low-fare itinerary associated with origin-destination pair (o, d)is $[1 - \lambda(t)] P_{od}$. In our test problems, we have $\lambda(t) = 0$ until about one third of the planning horizon, and after this, $\lambda(t)$ increases linearly to 1. Since the total expected demand for the capacity on flight leg i is $\sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} a_{ij} p_{jt}$, we measure the tightness of the leg capacities by

$$\theta = \frac{\sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{L}} a_{ij} \, p_{jt}}{\sum_{i \in \mathcal{L}} c_i}$$

We vary N, θ and κ in our computational experiments and label our test problems by (N, θ, κ) .

Section 3 shows that we can obtain an optimal solution to the problem $\min_{\alpha\geq 0} \{V_{1+k\tau/n}^{\alpha}(x_{1+k\tau/n})\}$ by solving a linear program. We use S = 50 for RLP. We use subgradient optimization to solve the problem $\min_{\beta\in\mathcal{F}}\{\sum_{i\in\mathcal{L}}\nu_{i,1+k\tau/n}^{\beta}(x_{i,1+k\tau/n})\}$ for DRA. The step size and termination criterion that we use for subgradient optimization are the same as those in Topaloglu (2009). When recomputing the bid prices for LAD, we solve the dual of problem (36)-(38) through column generation with 5% optimality gap. However, when computing the upper bounds on the optimal total expected revenue, we solve problem (36)-(38) to optimality so that the upper bounds that we report are accurate.

6.3 Computational Results for the First Airline Network

Tables 1 and 2 show the total expected revenues obtained by the different benchmark methods for the first airline network. Table 1 corresponds to the case where we recompute the bid prices five times over the planning horizon, whereas Table 2 corresponds to the case where we recompute the bid prices 50 times. The first column in these tables shows the characteristics of the test problems. The next eight columns show the total expected revenues obtained by LRD, LRDC, DLP, RLP, DPD, DRA, LAD

and LADC. We estimate these total expected revenues by simulating the performances of the different benchmark methods under multiple demand trajectories. To reduce the effect of simulation noise, we use common random numbers when simulating the performances of the different benchmark methods; see Law and Kelton (2000). The tenth column shows the percent gap between the total expected revenues obtained by LRDC and LRD. This column also includes a " \checkmark " whenever LRDC performs better than LRD, a "×" whenever LRDC performs worse than LRD and a " \odot " whenever there is no statistically significant difference between the total expected revenues obtained by LRDC and LRD at 95% level. The last six columns do the same thing as the tenth column, but they compare the performance of LRDC with the remaining six benchmark methods. LRDC turns out to be one of the better benchmark methods and we use it as a reference point.

The results indicate that LRDC performs noticeably better than the other benchmark methods. When we recompute the bid prices five times over the planning horizon, LRDC improves on LRD, DLP, RLP and LAD respectively by 5.6%, 7.5% 3.9% and 5.0% on the average. The same improvement figures decrease respectively to 2.5%, 4.1%, 1.5% and 2.9% when we recompute the bid prices 50 times. In either case, such improvement figures are considered quite significant in the network revenue management setting. The improvements of LRDC over LRD, DLP, RLP and LAD are most noticeable when the expected demand exceeds the capacity by a large margin and there is a large gap between the revenues associated with the high-fare and low-fare itineraries. Therefore, it is important to use a more sophisticated method to compute the bid prices when the leg capacities are scarce and there is substantial regret associated with accepting a low-fare itinerary request when one could have accepted a high-fare itinerary request.

LRD consistently performs better than DLP. This is due to the fact that problem (13)-(16) captures the dynamics of the network revenue management problem better than problem (25)-(28). On the other hand, randomization substantially improves the performance of the bid prices computed by problem (25)-(28) and RLP performs better than LRD and DLP for a majority of the test problems. Although both LRD and LAD compute dynamic bid prices, LAD has a slight advantage over LRD for a majority of the test problems.

The performances of LRDC, DPD, DRA and LADC are comparable. LRDC performs better than DPD and DRA by a small but consistent margin when we recompute the bid prices five times over the planning horizon. In this case, the total expected revenues obtained by LRDC are either better than or indistinguishable from those obtained by DPD and DRA. It turns out that DRA catches up with LRDC when we recompute the bid prices 50 times. On the other hand, LRDC generally performs better than DPD irrespective of how frequently we recompute the bid prices. This is expected since LRDC builds on the optimal dual solution to problem (13)-(16), whereas DPD builds on the optimal dual solution to problem (25)-(28) and problem (13)-(16) captures the capacity availabilities better than problem (25)-(28). The performances of LRDC and LADC are quite close. We note that LADC builds on the optimal solution to problem (36)-(38) and this problem provides a tighter upper bound on the optimal total expected revenue than problem (13)-(16). However, this potential advantage does not seem to help LADC too much and its performance is quite close to that of LRDC.

Figure 2 shows the improvements in the performances of the different benchmark methods when we recompute the bid prices 50 times over the planning horizon instead of five times. LRD, DLP, RLP and LAD significantly benefit from recomputing the bid prices more frequently. Recomputing the bid prices more frequently also helps the performance of DRA, but not as much as it helps the performances of LRD, DLP, RLP and LAD. On the other hand, LRDC, DPD and LADC do not benefit much from recomputing the bid prices more frequently. This observation can be explained by the fact that the decision rules used by LRD, DLP, RLP and LAD are equivalent to approximating the value functions by linear functions and the linear value function approximations tend to be accurate only around the current values of the remaining leg capacities, but not necessarily around the future values of the remaining leg capacities. In other words, the value function approximations used by LRD, DLP, RLP and LAD do not capture the curvature of the value functions and they need to be retuned frequently as the values of the remaining leg capacities change. On the other hand, the decision rules used by LRDC, DPD, DRA and LADC are equivalent to approximating the value functions by separable concave functions and these value function approximations capture the curvature of the value functions, at least to a certain extent. It is also interesting to note that there are quite a few test problems where the performances of LRDC, DPD and LADC slightly deteriorate when we recompute the bid prices more frequently. Similar behavior for other methods in the literature is analyzed in Cooper (2002).

Table 3 shows the CPU seconds required by the different benchmark methods to compute one set of bid prices. All of our computational experiments are run on a Pentium IV PC running Windows XP with 2.4 GHz CPU and 1 GB RAM. The runtimes for LAD and LADC correspond to the case where we solve problem (36)-(38) with 5% optimality gap. The results indicate that the runtimes for DLP, RLP and DPD are quite short. The runtimes for LRD are longer, but they are under two seconds for a majority of the test problems. The runtimes for LRD and LRDC differ by about one second. The runtimes for DRA exceed those for the other benchmark methods by orders of magnitude. Although DRA performs quite well when we recompute the bid prices 50 times over the planning horizon, this comes at the cost of a significant increase in the runtimes. The runtimes for LAD and LADC are also significantly longer than those for LRD and LRDC. Therefore, LRDC and DPD seem to strike a reasonable balance between performance and computational burden, though LRDC has a small but consistent performance advantage over DPD.

Table 4 shows the upper bounds on the optimal total expected revenues provided by the different benchmark methods. The columns in this table have the same interpretations as those in Tables 1 and 2, but they compare the upper bounds instead of the total expected revenues. The results indicate that DRA consistently provides the tightest upper bounds. The upper bounds provided by LRDC, RLP and LADC compete for the second place, but we emphasize that the upper bounds provided by RLP require estimating the expectation $\mathbb{E}\{\hat{Z}^1(D)\}$ through simulation, whereas the upper bounds provided by LRDC and LADC can be computed exactly. The upper bounds provided by LRD, DPD and LAD compete for the fifth place. DLP consistently provides the loosest upper bounds and this is in agreement with Propositions 5 and 7. One interesting observation is that LRD and LAD provide the same upper bounds for all of our test problems. Proposition 8 shows that the upper bound provided by LAD is potentially tighter than the one provided by LRD, but these upper bounds coincide for all of our test problems. We believe that this one of the reasons why LADC does not perform better than LRDC, although it is based on a linear program that potentially provides tighter upper bounds.

6.4 Computational Results for the Second Airline Network

Tables 5 and 6 show the total expected revenues obtained by the different benchmark methods for the second airline network. Table 5 corresponds to the case where we recompute the bid prices five times over the planning horizon, whereas Table 6 corresponds to the case where we recompute the bid prices 50 times. The columns in these tables have the same interpretations as those in Tables 1 and 2. For a majority of the test problems that take place over the second airline network, DRA requires more than 1 GB RAM and we do not use DRA as a benchmark method for the second airline network.

The results display essentially the same trends as those in Tables 1 and 2. When we recompute the bid prices five times over the planning horizon, LRDC improves on LRD, DLP, RLP and LAD respectively by 4.1%, 5.5%, 3.2% and 3.9% on the average. The same improvement figures decrease to 1.9%, 3.1%, 1.2% and 2.5% when we recompute the bid prices 50 times. LRDC generally performs better than DPD by a small but consistent margin. The performance gaps between LRDC and DPD are statistically significant for a majority of the test problems. The performance of LRDC is essentially indistinguishable from that of LADC.

Table 7 shows the CPU seconds required by the different benchmark methods to compute one set of bid prices. The runtimes for LRD are under six seconds and the runtimes for LRDC are under 10 seconds for a majority of the test problems. The runtimes for DPD are about twice as fast as those for LRDC. The runtimes for LAD and LADC approach one minute for the test problems with large numbers of spokes. Noting that LRDC performs essentially the same as LADC, LRDC appears to be an appealing alternative to LADC.

Table 8 shows the upper bounds on the optimal total expected revenues provided by the different benchmark methods. The columns in this table have the same interpretations as those in Table 4. The results indicate that LRDC, RLP and LADC provide the tightest upper bounds. The upper bounds provided by LRDC and LADC are tighter than those provided by RLP when the expected demand exceeds the capacity by a large margin. The upper bounds provided by LRD and LAD once again coincide for all of our test problems.

7 Conclusions

We presented a new method to compute bid prices in network revenue management problems. The novel aspect of our method is that it naturally provides dynamic bid prices that depend on how much time is left until departure. Our method provides an upper bound on the optimal total expected revenue and this upper bound is tighter than the one provided by the deterministic linear program. The bid prices computed by our method can be used in a dynamic programming decomposition-like idea to decompose the network revenue management problem by the flight legs and to obtain dynamic and capacity-dependent bid prices. Our computational experiments indicated that dynamic and capacity-dependent bid prices perform quite well. In particular, the four benchmark methods, LRDC, DPD, DRA and LADC, which compute dynamic and capacity-dependent bid prices, perform substantially better than LRD, DLP, RLP and LAD. Comparing LRDC, DPD, DRA and LADC among each other, DRA is comparable to LRDC when we recompute the bid prices frequently, but it has a higher computational burden. On the other hand, LRDC performs better than DPD by a small but consistent margin. The performance gaps between LRDC and DPD are statistically significant for many test problems and these performance gaps can approach 1% for the test problems with large numbers of spokes. The performances of LRDC and LADC are essentially indistinguishable, but the runtimes for LADC are significantly longer than those for LRDC. Therefore, LRDC, which uses the bid prices computed by our method as a starting point, seems to strike a reasonable balance between performance and computational burden.

References

Adelman, D. (2007), 'Dynamic bid-prices in revenue management', Operations Research 55(4), 647–661.

- Adelman, D. and Mersereau, A. J. (2008), 'Relaxations of weakly coupled stochastic dynamic programs', Operations Research 56(3), 712–727.
- Belobaba, P. P. (1987), Air Travel Demand and Airline Seat Inventory Control, PhD thesis, Massachusetts Institute of Technology, Cambridge, MA.
- Bertsekas, D. P. (2001), Dynamic Programming and Optimal Control, Athena Scientific, Belmont, MA.
- Bertsimas, D. and Popescu, I. (2003), 'Revenue management in a dynamic network environment', Transportation Science 37, 257–277.
- Castanon, D. A. (1997), Approximate dynamic programming for sensor management, in 'Proceedings of the 36th Conference on Decision & Control'.
- Cheung, R. K. and Powell, W. B. (1996), 'An algorithm for multistage dynamic networks with random arc capacities, with an application to dynamic fleet management', *Operations Research* 44(6), 951–963.
- Cooper, W. L. (2002), 'Asymptotic behavior of an allocation policy for revenue management', *Operations Research* **50**(4), 720–727.
- Gallego, G., Iyengar, G., Phillips, R. and Dubey, A. (2004), Managing flexible products on a network, CORC Technical Report TR-2004-01, Columbia University.
- Hawkins, J. (2003), A Lagrangian Decomposition Approach to Weakly Coupled Dynamic Optimization Problems and its Applications, PhD thesis, Massachusetts Institute of Technology, Cambridge, MA.
- Karmarkar, U. S. (1981), 'The multiperiod multilocation inventory problems', *Operations Research* 29, 215–228.
- Kunnumkal, S. and Topaloglu, H. (2007), A new dynamic programming decomposition method for the network revenue management problem with customer choice behavior, Technical report, Cornell University, School of Operations Research and Information Engineering. Available at http://legacy.orie.cornell.edu/~huseyin/publications/publications.html.
- Kunnumkal, S. and Topaloglu, H. (2008a), 'A duality-based relaxation and decomposition approach for inventory distribution systems', Naval Research Logistics Quarterly 55(7), 612–631.
- Kunnumkal, S. and Topaloglu, H. (2008b), 'A refined deterministic linear program for the network revenue management problem with customer choice behavior', Naval Research Logistics Quarterly 55, 563–580.

Law, A. L. and Kelton, W. D. (2000), Simulation Modeling and Analysis, McGraw-Hill, Boston, MA.

- Liu, Q. and van Ryzin, G. (2008), 'On the choice-based linear programming model for network revenue management', *Manufacturing & Service Operations Management* **10**(2), 288–310.
- Ruszczynski, A. (2003), Decomposition methods, in A. Ruszczynski and A. Shapiro, eds, 'Handbook in Operations Research and Management Science, Volume on Stochastic Programming', North Holland, Amsterdam.
- Simpson, R. W. (1989), Using network flow techniques to find shadow prices for market and seat inventory control, Technical report, MIT Flight Transportation Laboratory Memorandum M89-1, Cambridge, MA.
- Talluri, K. T. and van Ryzin, G. J. (2004), *The Theory and Practice of Revenue Management*, Kluwer Academic Publishers.
- Talluri, K. and van Ryzin, G. (1998), 'An analysis of bid-price controls for network revenue management', Management Science 44(11), 1577–1593.
- Talluri, K. and van Ryzin, G. (1999), 'A randomized linear programming method for computing network bid prices', *Transportation Science* **33**(2), 207–216.
- Topaloglu, H. (2009), 'Using Lagrangian relaxation to compute capacity-dependent bid-prices in network revenue management', *Operations Research* 57(3), 637–649.
- Topaloglu, H. and Kunnumkal, S. (2006), Computing time-dependent bid prices in network revenue management problems, Technical report, Cornell University, School of Operations Research and Information Engineering. Available at http://legacy.orie.cornell.edu/~huseyin/publications/publications.html.
- Varaiya, P. P. (1998), 'Lecture notes on optimization', Unpublished manuscript, University of California, Department of Electrical Engineering and Computer Science.
- Williamson, E. L. (1992), Airline Network Seat Control, PhD thesis, Massachusetts Institute of Technology, Cambridge, MA.
- Wolsey, L. A. (1998), Integer Programming, John Wiley & Sons, Inc., New York.
- Zhang, D. and Adelman, D. (2006), An approximate dynamic programming approach to network revenue management with customer choice, Technical report, University of Chicago, Graduate School of Business.



Figure 1: Structures of the two airline networks that we use in our computational experiments.



Figure 2: Improvements in the performances of the benchmark methods for the first airline network when we recompute the bid prices 50 times over the planning horizon instead of five times.

	LADC	$0.01 \odot$	-0.04 \odot	-0.05 \odot	$0.14 \odot$	-0.02 \odot	$0.04 \odot$	$0.14 \odot$	$0.06 \odot$	$0.02 \odot$	$0.02 \odot$	$0.03 \odot$	$0.01 \odot$	$0.04 \odot$	$0.15 \checkmark$	-0.09 ⊙	$0.02 \odot$	$\odot 60.0$	-0.01 \odot	-0.02 ⊙	$0.01 \odot$	$0.01 \odot$	$0.15 \odot$	-0.10 \odot	-0.01 \odot	$0.15 \odot$	$\odot 60.0$	-0.05 ⊙	0.03
	LAD	$1.44 \checkmark$	$2.08 \checkmark$	$4.09 \checkmark$	$2.74 \checkmark$	$3.69 \checkmark$	7.11 🗸	$3.15 \checkmark$	$8.08 \checkmark$	7.95 🗸	$2.46 \checkmark$	$2.96 \checkmark$	$3.72 \checkmark$	4.20 \checkmark	$4.73 \checkmark$	$6.58 \checkmark$	$5.35 \checkmark$	7.37 🗸	$9.35 \checkmark$	2.37 🗸	$3.76 \checkmark$	$4.96 \checkmark$	$3.25 \checkmark$	5.18 \checkmark	$7.16 \checkmark$	$4.74 \checkmark$	7.07 🗸	9.29 🗸	4.99
DC and	DRA	$0.16 \odot$	$0.33\odot$	$1.00 \checkmark$	$0.27 \odot$	$0.78 \checkmark$	$1.29 \checkmark$	-0.11 \odot	$0.69 \odot$	$1.15 \checkmark$	$0.29 \odot$	0.44 \odot	$0.49 \odot$	$0.51 \checkmark$	$0.53 \checkmark$	$0.55\odot$	$0.20 \odot$	$1.20 \checkmark$	$1.20 \checkmark$	$0.07 \odot$	0.40 \odot	$0.53\odot$	0.42 \checkmark	$0.56\odot$	$0.58\odot$	$0.20 \odot$	$0.42 \odot$	$0.47 \odot$	0.54
tween LR.	DPD	0.22 🗸	$0.03 \odot$	- 0.02 \odot	$0.22 \checkmark$	$0.04 \odot$	$0.07 \odot$	$0.22\odot$	$0.07 \odot$	$0.10 \odot$	0.13 \odot	$0.16 \checkmark$	$0.15 \checkmark$	$0.19 \odot$	$0.36 \checkmark$	$0.28 \checkmark$	$0.18 \odot$	$0.39 \checkmark$	$0.22\odot$	0.46 🗸	$0.58 \checkmark$	$0.32 \checkmark$	$0.28 \checkmark$	$0.17 \odot$	$0.38 \checkmark$	$0.45 \checkmark$	$0.32 \odot$	0.27 \checkmark	0.23
% gap be	RLP	1.31 🗸	$2.40 \checkmark$	$3.59 \checkmark$	$2.14 \checkmark$	$3.69 \checkmark$	$5.28 \checkmark$	$2.16 \checkmark$	$4.73 \checkmark$	$5.73 \checkmark$	$1.85 \checkmark$	$2.97 \checkmark$	$4.14 \checkmark$	$2.41 \checkmark$	$4.34 \checkmark$	$5.22 \checkmark$	$3.38 \checkmark$	$5.68 \checkmark$	$4.85 \checkmark$	$1.79 \checkmark$	$3.45 \checkmark$	$4.65 \checkmark$	$2.74 \checkmark$	$4.96 \checkmark$	$5.76 \checkmark$	$3.71 \checkmark$	$5.70 \checkmark$	$5.95 \checkmark$	3.87
	DLP	$1.23 \checkmark$	$3.21 \checkmark$	$6.05 \checkmark$	$2.44 \checkmark$	$6.51 \checkmark$	10.85 \checkmark	$4.15 \checkmark$	10.40 \checkmark	$16.51 \checkmark$	2.65 √	$4.90 \checkmark$	$7.49 \checkmark$	$3.84 \checkmark$	$8.12 \checkmark$	$12.79 \checkmark$	$4.77 \checkmark$	$10.48 \checkmark$	$16.72 \checkmark$	2.73 🗸	$5.47 \checkmark$	8.85 🗸	$3.08 \checkmark$	$7.51 \checkmark$	$12.58 \checkmark$	$3.79 \checkmark$	$9.16 \checkmark$	15.33 \checkmark	7.47
	LRD	$1.10 \checkmark$	$2.52 \checkmark$	$4.83 \checkmark$	$2.09 \checkmark$	$5.60 \checkmark$	$9.15 \checkmark$	$3.86 \checkmark$	8.76 🗸	$12.70 \checkmark$	2.24 🗸	$3.67 \checkmark$	$5.56 \checkmark$	2.99 \checkmark	$5.79 \checkmark$	8.38 🗸	$3.87 \checkmark$	8.77 🗸	11.61 \checkmark	2.05 🗸	$3.97 \checkmark$	$6.51~\checkmark$	$2.60 \checkmark$	$5.09 \checkmark$	8.41 🗸	$2.88 \checkmark$	$6.04 \checkmark$	$9.92 \checkmark$	5.59
	LADC	11,890	16,968	27,425	10,649	15,708	25,979	8,808	13,792	24,039	13,795	19,833	32,176	12,544	18,395	30,680	10,346	16,087	28, 220	14,673	20,931	33,926	13,291	19,431	32,300	10,936	16,843	29,566	
	LAD	11,720	16,608	26,292	10,371	15,127	24,142	8,542	12,685	22,132	13,458	19,252	30,984	12,021	17,551	28,636	9,795	14,915	25,579	14,322	20,146	32,247	12,878	18,408	29,983	10,433	15,666	26,807	
ained by	DRA	11,872	16,905	27,138	10,635	15,583	25,653	8,830	13,705	23,767	13,757	19,751	32,023	12,485	18, 326	30,483	10,327	15,908	27,879	14,660	20,849	33,750	13,255	19,303	32,110	10,930	16,787	29,412	
venue obt	DPD	11,865	16,956	27,418	10,640	15,699	25,969	8,801	13,791	24,019	13,780	19,808	32,131	12,525	18,356	30,565	10,330	16,038	28,156	14,602	20,812	33,821	13,273	19,380	32,172	10,903	16,805	29,472	
spected re	RLP	11,736	16,554	26,429	$10,\!436$	15,126	24,616	8,630	13,148	22,666	13,543	19,251	30,847	$12,\!247$	17,624	29,052	9,998	15,187	26,849	14,408	20,211	32, 353	12,946	$18,\!451$	30,435	10,546	15,896	27,794	
Total ex	DLP	11,745	16,417	25,754	10,404	14,683	23,169	8,454	12,365	20,073	13,432	18,867	29,769	12,067	16,927	26,730	9,855	14,414	23,499	14,270	19,789	30,925	12,901	17,955	28,233	10,537	15,313	25,021	
	LRDC	11,891	16,961	27,412	10,664	15,706	25,988	8,820	13,800	24,044	13,797	19,840	32,180	12,549	18,423	30,652	10,348	16,101	28,217	14,670	20,934	33,929	13,311	19,413	32,296	10,952	16,858	29,552	
	LRD	11,761	16,534	26,089	10,441	14,826	23,611	8,479	12,592	20,990	13,488	19,111	30, 391	12,174	17,356	28,085	9,947	14,689	24,940	14,370	20,102	31,722	12,964	18,425	29,580	10,637	15,839	26,622	
Problem	(N,θ,κ)	(4, 1.0, 2)	(4, 1.0, 4)	(4, 1.0, 8)	(4, 1.2, 2)	(4, 1.2, 4)	(4, 1.2, 8)	(4, 1.6, 2)	(4, 1.6, 4)	(4, 1.6, 8)	(6, 1.0, 2)	(6, 1.0, 4)	(6, 1.0, 8)	(6, 1.2, 2)	(6, 1.2, 4)	(6, 1.2, 8)	(6, 1.6, 2)	(6, 1.6, 4)	(6, 1.6, 8)	(8, 1.0, 2)	(8, 1.0, 4)	(8, 1.0, 8)	(8, 1.2, 2)	(8, 1.2, 4)	(8, 1.2, 8)	(8, 1.6, 2)	(8, 1.6, 4)	(8, 1.6, 8)	Average
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Table 1: Total expected revenues obtained by the benchmark methods for the first airline network when we recompute the bid prices five times over the planning horizon.

Problem Problem Main Functional systemet resume observation by many systemestic lysemic lys																														
Problem Total expected transme obtained by $\gamma_{\rm exp}$ gas <t< td=""><td></td><td>LADC</td><td>0.02 \odot</td><td>0.01 \odot</td><td>-0.04 \odot</td><td>-0.02 \odot</td><td>0.07 \odot</td><td>$0.02 \checkmark$</td><td>0.11 \odot</td><td>0.10 \odot</td><td>-0.01 \odot</td><td>0.06</td><td>$0.01 \odot$</td><td>$0.02~\odot$</td><td>0.13 \odot</td><td>-0.01 ①</td><td>$0.01 \odot$</td><td>0.01 \odot</td><td>$0.23 \checkmark$</td><td>0.00 \odot</td><td>$0.10 \odot$</td><td>0.00 \odot</td><td>0.06 \odot</td><td>$0.24 \checkmark$</td><td>-0.03 \odot</td><td>$0.01 \odot$</td><td>$0.28 \odot$</td><td>$0.12 \odot$</td><td>-0.02 \odot</td><td>0.05</td></t<>		LADC	0.02 \odot	0.01 \odot	-0.04 \odot	-0.02 \odot	0.07 \odot	$0.02 \checkmark$	0.11 \odot	0.10 \odot	-0.01 \odot	0.06	$0.01 \odot$	$0.02~\odot$	0.13 \odot	-0.01 ①	$0.01 \odot$	0.01 \odot	$0.23 \checkmark$	0.00 \odot	$0.10 \odot$	0.00 \odot	0.06 \odot	$0.24 \checkmark$	-0.03 \odot	$0.01 \odot$	$0.28 \odot$	$0.12 \odot$	-0.02 \odot	0.05
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $		LAD	0.83 🗸	$1.33 \checkmark$	$2.90 \checkmark$	$0.92 \checkmark$	$2.07 \checkmark$	2.92 \checkmark	$1.72 \checkmark$	$3.27 \checkmark$	$5.28 \checkmark$	$1.52 \checkmark$	$1.88 \checkmark$	$2.79 \checkmark$	$1.87 \checkmark$	$2.37 \checkmark$	$3.63 \checkmark$	$2.37 \checkmark$	$3.99 \checkmark$	$5.70 \checkmark$	$1.25 \checkmark$	$1.92 \checkmark$	$3.49 \checkmark$	$2.17 \checkmark$	$2.93 \checkmark$	$4.26 \checkmark$	$2.45 \checkmark$	$4.04 \checkmark$	$6.99 \checkmark$	2.85
Chrolem Total expected revenue obtained by ADD LAD LAD DLP RLP DPD (N, θ, κ) LRD D12 RLP DPD DRA LAD LAD DLP RLP DPD $(4, 10, 4)$ 16,816 16,825 16,705 16,823 16,903 16,756 15,87 133 0.09 0.110 0.110 $(4, 10, 4)$ 16,816 16,825 16,705 16,823 16,903 16,676 15,87 10.85 0.30 0.110 0.01	DC and	DRA	-0.08 ⊙	$0.19 \odot$	$0.55 \checkmark$	-0.12 \odot	$0.55 \checkmark$	$0.16 \checkmark$	-0.31 \times	0.40 \odot	$0.10 \checkmark$	-0.03 ⊙	-0.22 ①	-0.22 \odot	$0.07 \odot$	-0.49 \times	-0.16 \odot	$0.14 \odot$	0.33 \odot	0.33 \odot	$-0.31 \times$	-0.23 \odot	-0.02 \odot	0.03 \odot	-0.02 \odot	-0.19 \odot	-0.16 \odot	$0.05 \odot$	0.05 \odot	0.01
Problem Total expected revenue obtained by ADD IRD Number of the system of the	etween LR	DPD	$0.08 \odot$	0.11 \odot	$0.08 \odot$	$0.02~\odot$	-0.02 \odot	$0.16 \checkmark$	$0.07 \odot$	$0.24 \odot$	$0.25 \checkmark$	0.33 🗸	0.32 \checkmark	$0.16 \checkmark$	$0.38 \checkmark$	$0.33 \checkmark$	$0.22 \checkmark$	$0.23~\odot$	$0.44 \checkmark$	$0.48 \checkmark$	$0.36 \checkmark$	$0.92 \checkmark$	$0.26 \checkmark$	$0.56 \checkmark$	$0.80 \checkmark$	$0.50 \checkmark$	$0.61 \checkmark$	$0.87 \checkmark$	$0.50 \checkmark$	0.34
Problem Total expected revenue obtained by LAD LADC DLP (N, θ, κ) LRD I.RDC DLP RLP DPA LAD LAD DLP $(4, 10, 2)$ 11,876 11,876 11,876 11,896 11,876 0.16 0.440 $(4, 10, 2)$ 16,816 16,993 16,931 16,932 0.96 0.16 0.440 $(4, 10, 2)$ 16,816 16,993 16,942 27,266 26,623 27,428 3.62 0.440 $(4, 12, 2)$ 16,559 15,644 15,594 16,651 0.20 0.51 0.43 $(4, 12, 3)$ 13,312 13,574 13,573 13,573 13,573 13,773 13,774 13,770 13,77 14,05 $(4, 16, 2)$ 13,317 13,573 13,573 13,773 13,773 13,773 13,77 14,05 $(4, 16, 2)$ 13,374 13,570 13,773 13,773 13,774 14,07 6 14,05 14,05 14,05	% gap be	RLP	$0.52 \checkmark$	$0.90 \checkmark$	$1.23 \checkmark$	$0.39 \checkmark$	$1.66 \checkmark$	$1.92 \checkmark$	$0.74 \checkmark$	$1.79 \checkmark$	$1.68 \checkmark$	0.77 🗸	$0.93 \checkmark$	$0.64 \odot$	$1.07 \checkmark$	$1.45 \checkmark$	$1.69 \checkmark$	$1.19 \checkmark$	$2.24 \checkmark$	$2.66 \checkmark$	0.53 🗸	$1.31 \checkmark$	$1.87 \checkmark$	$1.31 \checkmark$	$2.09 \checkmark$	$2.66 \checkmark$	$1.55 \checkmark$	$2.73 \checkmark$	$2.84 \checkmark$	1.50
		DLP	$0.40 \checkmark$	$1.63 \checkmark$	$3.62 \checkmark$	$0.51 \checkmark$	$3.16 \checkmark$	$6.32 \checkmark$	$1.46 \checkmark$	$4.92 \checkmark$	$9.00 \checkmark$	$1.40 \checkmark$	$2.88 \checkmark$	$4.54 \checkmark$	$1.69 \checkmark$	$4.24 \checkmark$	$7.25 \checkmark$	$2.19 \checkmark$	$6.38 \checkmark$	$10.07 \checkmark$	0.70 🗸	$2.57 \checkmark$	$5.25 \checkmark$	$1.62 \checkmark$	$4.05 \checkmark$	7.85 🗸	$1.74 \checkmark$	$5.57 \checkmark$	10.61 \checkmark	4.13
Total expected revenue obtained by Total expected revenue obtained by (N, θ, κ) LRD LRDC DLP RLP DPD DRA LAD LAD LAD $(4, 1.0, 2)$ 11,876 11,892 16,982 16,756 15,949 16,756 11,892 $(4, 1.0, 2)$ 16,816 15,923 16,933 15,944 15,764 15,933 15,944 $(4, 1.2, 3)$ 15,354 15,704 15,208 15,444 15,708 15,694 $(4, 1.2, 3)$ 15,354 15,704 15,208 15,618 15,530 15,694 $(4, 1.2, 3)$ 23,712 13,804 13,771 13,771 13,773 13,733 13,733 13,733 13,733 13,730 13,731 13,733 13,770 13,733 13,770		LRD	$0.16 \odot$	$0.98 \checkmark$	$2.38 \checkmark$	0.20 \odot	$2.23 \checkmark$	$4.62 \checkmark$	$0.83 \checkmark$	$3.57 \checkmark$	6.27 \checkmark	0.77 ✓	$1.72 \checkmark$	$2.95 \checkmark$	$0.95 \checkmark$	$2.30 \checkmark$	$4.06 \checkmark$	$1.37 \checkmark$	$3.81 \checkmark$	$5.91 \checkmark$	0.60 ✓	$1.16 \checkmark$	2.92 \checkmark	$1.09 \checkmark$	$2.40 \checkmark$	$4.66 \checkmark$	$1.10 \checkmark$	$2.69 \checkmark$	$6.01 \checkmark$	2.51
Problem Total expected revenue obtained by (N, θ, κ) LRD LRDC DLP RLP DPD DRA LAD (4, 1.0, 2) 11,876 11,895 11,847 11,832 11,836 11,904 16,756 (4, 1.0, 8) 26,765 27,418 26,426 27,060 27,266 26,653 (4, 1.2, 2) 10,627 10,649 10,594 16,756 27,418 26,652 (4, 1.2, 2) 10,627 10,649 10,551 10,662 10,561 15,538 (4, 1.2, 8) 26,756 27,418 26,426 27,066 21,653 13,771 (4, 1.6, 1) 21 33,312 13,312 13,414 15,704 13,550 (4, 1.6, 1) 21,3312 13,657 13,673 13,771 13,733 13,570 (4, 1.6, 1) 19,475 13,887 8,773 33,571 13,570 13,232 (6, 1.0, 4) 19,475 13,123 13,123 13,771 13,771 13,733 1		LADC	11,892	16,980	27,428	10,651	15,694	25,959	8,806	13,790	23,986	13,772	19,813	32,121	12,524	18,344	30,564	10,349	16,019	28,171	14,638	20,894	33,868	13,278	19,389	32,195	10,924	16,800	29,450	
ProblemTotal expected revenue obtained by (N, θ, κ)IRDIRDCDLPBLPDPDDRA(4,110, 2)11,87611,89511,84711,83211,83611,904(4,110, 4)16,81616,98216,70516,82816,96316,949(4,112, 4)16,81616,98216,70516,82816,94710,662(4,112, 4)15,35415,70415,20815,44415,70815,618(4,112, 8)24,76625,96624,32425,46825,92425,926(4,112, 8)27,48023,98513,12413,72413,749(4,116, 8)22,48023,98521,82523,55223,92623,960(5,10, 4)19,47519,47519,47519,47519,45713,73413,734(6,10, 2)13,13132,12830,67731,92332,70032,3960(6,10, 4)19,47519,47519,53713,77113,73413,734(5,10, 8)31,18132,12830,67731,92332,700(6,10, 4)19,47519,53713,77113,73413,736(6,10, 4)19,47519,53713,77113,73413,734(6,10, 4)19,47519,53713,73413,734(7,10, 8)31,18132,12830,67731,92313,436(6,10, 4)19,47519,376612,49312,63616,693(6,10, 4)19,47519,376612,49312,636(6,10, 4		LAD	11,796	16,756	26,623	10,551	15,380	25,207	8,663	13,353	22,718	13,570	19,443	31,232	12,306	17,907	29,457	10,104	15,415	26,566	14,468	20,493	32,705	13,022	18,815	30,825	10,686	16,141	27,388	
ProblemTotal expected revenue obt (N, θ, κ) LRDLRDCDtal expected revenue obt (N, θ, κ) (N, θ, κ) LRDLRDCDLPRLPDPD $(4, 10, 4)$ 16,81616,98216,70516,82816,963 $(4, 10, 4)$ 16,81616,98216,70516,82816,963 $(4, 1.2, 2)$ 10,62710,64910,59410,60810,647 $(4, 1.2, 3)$ 10,62710,64910,59410,60810,647 $(4, 1.2, 3)$ 13,31213,80413,12415,70827,396 $(4, 1.6, 2)$ 13,31213,80413,12415,70827,930 $(4, 1.6, 2)$ 13,31213,80413,12415,70810,647 $(4, 1.6, 8)$ 22,48023,98521,82523,92677,713 $(6, 1.0, 2)$ 13,77113,22523,59223,926 $(6, 1.0, 4)$ 19,47519,81519,24419,63119,751 $(6, 1.2, 2)$ 12,40317,56518,07718,282 $(6, 1.2, 4)$ 17,92118,34317,56518,07718,282 $(6, 1.2, 8)$ 29,32730,56728,34930,05030,499 $(6, 1.2, 4)$ 19,65512,40612,493(710 $(6, 1.2, 4)$ 17,92118,34416,05513,19219,751 $(6, 1.2, 2)$ 19,37713,23210,12310,22710,326 $(6, 1.2, 4)$ 10,20810,12310,22710,32616,963 $(6, 1.2, 4)$ 15,444 <t< td=""><td>tained by</td><td>DRA</td><td>11,904</td><td>16,949</td><td>27,266</td><td>10,662</td><td>15,618</td><td>25,923</td><td>8,843</td><td>13,749</td><td>23,960</td><td>13,783</td><td>19,859</td><td>32,200</td><td>12,533</td><td>18,432</td><td>30,616</td><td>10,335</td><td>16,002</td><td>28,078</td><td>14,698</td><td>20,941</td><td>33,894</td><td>13,306</td><td>19,386</td><td>32,260</td><td>10,972</td><td>16,811</td><td>29,429</td><td></td></t<>	tained by	DRA	11,904	16,949	27,266	10,662	15,618	25,923	8,843	13,749	23,960	13,783	19,859	32,200	12,533	18,432	30,616	10,335	16,002	28,078	14,698	20,941	33,894	13,306	19,386	32,260	10,972	16,811	29,429	
ProblemTotal expected re (N, θ, κ) LRDLRDCDLPRLP (N, θ, κ) LRDII,87611,84711,832 $(4, 1.0, 4)$ 16,81616,98216,70516,828 $(4, 1.0, 8)$ 26,76527,41826,42627,080 $(4, 1.2, 2)$ 10,62710,64910,59410,608 $(4, 1.2, 3)$ 15,35415,70415,20815,444 $(4, 1.2, 8)$ 24,76625,96624,32425,468 $(4, 1.2, 8)$ 24,76625,96624,32413,558 $(4, 1.6, 8)$ 13,31213,80113,12313,558 $(4, 1.6, 8)$ 13,31213,80321,82523,562 $(4, 1.6, 8)$ 22,48023,98521,82523,605 $(6, 1.0, 4)$ 19,47519,81519,24419,631 $(6, 1.2, 8)$ 21,22113,57310,227 $(6, 1.2, 8)$ 23,18132,132327,420 $(6, 1.2, 8)$ 23,23730,56728,34930,050 $(6, 1.2, 8)$ 29,32730,56728,34930,050 $(6, 1.2, 8)$ 29,32730,56728,34930,727 $(6, 1.2, 8)$ 29,32730,56728,17025,33327,420 $(6, 1.2, 8)$ 29,50628,17025,33327,420 $(6, 1.2, 4)$ 15,74416,05511,5,696(6,1.2,8)10,227 $(6, 1.2, 8)$ 29,50628,17025,33327,420 $(6, 1.2, 4)$ 15,74416,05510,12310,227	venue obt	DPD	11,886	16,963	27,396	10,647	15,708	25,924	8,809	13,771	23,926	13,734	19,751	32,077	12,493	18,282	30,499	10,326	15,984	28,035	14,599	20,701	33,800	13,236	19,228	32,037	10,888	16,673	29, 298	
ProblemTotal ex (N, θ, κ) LRDLRDDrotal ex (N, θ, κ) LRDLRDCDLP $(4, 1.0, 2)$ 11,87611,89511,847 $(4, 1.0, 8)$ 26,76527,41826,426 $(4, 1.2, 2)$ 10,62710,64910,594 $(4, 1.2, 8)$ 26,76527,41826,426 $(4, 1.2, 8)$ 24,76625,96624,324 $(4, 1.6, 8)$ 22,48023,98521,825 $(4, 1.6, 8)$ 22,48023,98521,825 $(6, 1.0, 4)$ 19,47519,81519,244 $(6, 1.0, 8)$ 31,18132,12830,567 $(6, 1.2, 8)$ 29,32730,56728,349 $(6, 1.2, 8)$ 29,32730,56728,349 /i $(6, 1.2, 8)$ 10,20810,35010,123 $(6, 1.6, 8)$ 29,32730,56728,349 $(6, 1.6, 8)$ 29,32730,56728,349 $(6, 1.6, 8)$ 29,32730,56728,349 $(6, 1.6, 8)$ 29,32730,56728,333 $(8, 1.0, 4)$ 15,44416,05510,123 $(8, 1.2, 2)$ 14,56514,56514,565 $(8, 1.2, 2)$ 13,16513,31013,095 $(8, 1.6, 3)$ 20,69832,109 $(8, 1.6, 4)$ 18,91819,38318,599 $(8, 1.6, 4)$ 16,36816,82015,883 $(8, 1.6, 4)$ 16,36810,95510,764 $(8, 1.6, 4)$ 16,36810,95510,764 $(8, 1.6, 4)$ 16,368 <td< td=""><td>pected re</td><td>RLP</td><td>11,832</td><td>16,828</td><td>27,080</td><td>10,608</td><td>15,444</td><td>25,468</td><td>8,750</td><td>13,558</td><td>23,582</td><td>13,673</td><td>19,631</td><td>31,923</td><td>12,406</td><td>18,077</td><td>30,050</td><td>10,227</td><td>15,696</td><td>27,420</td><td>14,574</td><td>20,618</td><td>33,254</td><td>13, 136</td><td>18,978</td><td>31, 342</td><td>10,785</td><td>16,360</td><td>28,608</td><td></td></td<>	pected re	RLP	11,832	16,828	27,080	10,608	15,444	25,468	8,750	13,558	23,582	13,673	19,631	31,923	12,406	18,077	30,050	10,227	15,696	27,420	14,574	20,618	33,254	13, 136	18,978	31, 342	10,785	16,360	28,608	
ProblemIRDLRDLRDC (N, θ, κ) LRDLRDLRDC $(4, 1.0, 2)$ 11,87611,895 $(4, 1.0, 8)$ 16,81616,982 $(4, 1.0, 8)$ 16,81616,982 $(4, 1.2, 2)$ 10,62710,649 $(4, 1.2, 8)$ 26,76527,418 $(4, 1.2, 8)$ 24,76625,966 $(4, 1.2, 8)$ 24,76625,966 $(4, 1.6, 8)$ 22,48023,8315 $(6, 1.0, 8)$ 31,18132,128 $(6, 1.0, 8)$ 31,18132,128 $(6, 1.2, 2)$ 13,67413,779 $(6, 1.2, 2)$ 13,67413,779 $(6, 1.2, 8)$ 22,48023,985 $(6, 1.2, 8)$ 22,44116,055 $(6, 1.2, 8)$ 22,42023,935 $(6, 1.2, 8)$ 20,32730,567 $(6, 1.2, 8)$ 20,64920,893 $(8, 1.0, 2)$ 14,56514,652 $(8, 1.0, 8)$ 32,90033,888 $(8, 1.2, 8)$ 20,64920,893 $(8, 1.2, 8)$ 20,64920,893 $(8, 1.2, 8)$ 20,63628,170 $(8, 1.2, 8)$ 20,63810,955 $(8, 1.6, 8)$ 20,63810,955 $(8, 1.6, 8)$ 27,67729,445 $(8, 1.6, 8)$ 27,67729,445	Total ex	DLP	11,847	16,705	26,426	10,594	15,208	24, 324	8,687	13, 124	21,825	13,587	19,244	30,671	12, 329	17,565	28, 349	10,123	15,031	25,333	14,549	20,356	32,109	13,095	18,599	29,672	10,764	15,883	26, 322	
ProblemProblem (N, θ, κ) (N, θ, κ) (N, θ, κ) $(1, 0, 2)$ $(1, 10, 2)$ $(1, 10, 8)$ $(2, 11, 2)$ $(1, 10, 8)$ $(2, 11, 2)$ $(1, 10, 8)$ $(2, 11, 2)$ $(2, 11, 2)$ $(3, 11, 2)$ $(4, 11, 2)$ $(4, 11, 2)$ $(4, 11, 2)$ $(4, 11, 2)$ $(4, 11, 2)$ $(4, 11, 2)$ $(4, 11, 2)$ $(4, 11, 2)$ $(4, 11, 2)$ $(4, 11, 2)$ $(4, 11, 2)$ $(4, 11, 2)$ $(4, 11, 2)$ $(4, 11, 2)$ $(5, 11, 2)$ $(1, 2, 2)$ $(1, 2, 2)$ $(1, 2, 2)$ $(1, 2, 2)$ $(1, 2, 3)$ $(2, 10, 2)$ $(3, 10, 2)$ $(3, 10, 2)$ $(3, 10, 2)$ $(3, 10, 3)$ $(3, 10, 6)$ $(3, 10, 6)$ $(3, 10, 8)$ $(3, 10, 8)$ $(3, 10, 8)$ $(3, 10, 8)$ $(3, 10, 8)$ $(3, 10, 8)$ $(3, 10, 8)$ $(3, 10, 8)$ $(3, 10, 8)$ $(3, 10, 6)$ $(3, 10, 8)$ $(3, 10, 8)$ $(3, 10, 8)$ $(3, 10, 8)$ $(3, 10, 8)$ $(3, 10, 8)$ $(3, 10, 8)$ $(3, 10, 8)$ $(3, 10, 8)$ $(3, 10, 8)$ $(3, 10, 8)$ $(3, 10, 8)$ $(3, 10, 8)$ $(3, 10, 8)$ $(3, 10, 8)$ $(4, 10, 8)$ $(4, 10,$		LRDC	11,895	16,982	27,418	10,649	15,704	25,966	8,815	13,804	23,985	13,779	19,815	32,128	12,541	18,343	30,567	10,350	16,055	28,170	14,652	20,893	33,888	13, 310	19,383	32,198	10,955	16,820	29,445	
$\begin{array}{c} Problem \\ \hline (N,\theta,\kappa) \\ \hline (N,\theta,\kappa) \\ \hline (A,1.0,2) \\ (4,1.0,4) \\ (4,1.2,2) \\ (4,1.2,2) \\ (4,1.2,4) \\ (4,1.2,4) \\ (4,1.2,4) \\ (6,1.0,4) \\ (6,1.0,8) \\ (6,1.2,4) \\ (6,1.0,8) \\ (6,1.2,4) \\ (6,1.0,8) \\ (6,1.2,4) \\ (6,1.0,8) \\ (6,1.2,4) \\ (6,1.0,8) \\ (6,1.2,2) \\ (8,1.2,2) \\ (1,1,2,2) \\$		LRD	11,876	16,816	26,765	10,627	15,354	24,766	8,742	13,312	22,480	13,674	19,475	31,181	12,421	17,921	29,327	10,208	15,444	26,506	14,565	20,649	32,900	13,165	18,918	30,698	10,834	16,368	27,677	
	Problem	(N,θ,κ)	(4, 1.0, 2)	(4, 1.0, 4)	(4, 1.0, 8)	(4, 1.2, 2)	(4, 1.2, 4)	(4, 1.2, 8)	(4, 1.6, 2)	(4, 1.6, 4)	(4, 1.6, 8)	(6, 1.0, 2)	(6, 1.0, 4)	(6, 1.0, 8)	(6, 1.2, 2)	(6, 1.2, 4)	(6, 1.2, 8)	(6, 1.6, 2)	(6, 1.6, 4)	(6, 1.6, 8)	(8, 1.0, 2)	(8, 1.0, 4)	(8, 1.0, 8)	(8, 1.2, 2)	(8, 1.2, 4)	(8, 1.2, 8)	(8, 1.6, 2)	(8, 1.6, 4)	(8, 1.6, 8)	Average

Table 2: Total expected revenues obtained by the benchmark methods for the first airline network when we recompute the bid prices 50 times over the planning horizon.

Problem				CPU se	econds fo	or		
(N, θ, κ)	LRD	LRDC	DLP	RLP	DPD	DRA	LAD	LADC
(4, 1.0, 2)	0.31	0.81	0.00	0.03	0.42	76.38	6.25	7.01
(4, 1.0, 4)	0.24	0.81	0.02	0.05	0.42	39.94	6.72	6.92
(4, 1.0, 8)	0.23	0.82	0.02	0.05	0.42	13.63	6.42	7.58
(4, 1.2, 2)	0.33	0.80	0.02	0.05	0.34	30.56	8.13	8.66
(4, 1.2, 4)	0.24	0.73	0.00	0.05	0.34	57.81	7.20	8.30
(4, 1.2, 8)	0.24	0.72	0.00	0.05	0.36	60.47	7.50	7.75
(4, 1.6, 2)	0.41	0.77	0.00	0.05	0.28	44.11	8.81	9.26
(4, 1.6, 4)	0.26	0.63	0.00	0.05	0.28	32.13	8.71	8.95
(4, 1.6, 8)	0.24	0.61	0.00	0.05	0.28	51.31	8.17	8.80
(6, 1.0, 2)	1.23	2.53	0.00	0.06	0.92	52.19	15.58	17.05
(6, 1.0, 4)	1.09	2.41	0.00	0.06	0.92	34.02	13.74	16.39
(6, 1.0, 8)	1.06	2.37	0.00	0.06	0.92	33.45	16.60	17.89
(6, 1.2, 2)	1.81	2.87	0.00	0.06	0.78	82.09	24.31	25.67
(6, 1.2, 4)	1.30	2.41	0.00	0.06	0.78	234.86	30.28	32.50
(6, 1.2, 8)	1.13	2.22	0.00	0.06	0.78	231.20	26.45	27.63
(6, 1.6, 2)	3.12	3.87	0.00	0.06	0.59	45.28	27.81	28.59
(6, 1.6, 4)	1.65	2.49	0.00	0.08	0.59	165.77	33.52	35.25
(6, 1.6, 8)	1.21	2.06	0.00	0.06	0.59	119.44	29.05	30.86
(8, 1.0, 2)	1.02	3.30	0.02	0.09	1.66	144.61	33.09	36.66
(8, 1.0, 4)	0.60	2.91	0.00	0.09	1.63	360.86	36.95	39.55
(8, 1.0, 8)	0.47	2.81	0.02	0.09	1.63	307.61	30.69	33.64
(8, 1.2, 2)	3.30	5.19	0.00	0.11	1.39	129.72	64.06	66.72
(8, 1.2, 4)	1.00	2.95	0.02	0.11	1.39	284.78	57.49	59.91
(8, 1.2, 8)	0.54	2.55	0.02	0.09	1.41	206.91	52.33	54.95
(8, 1.6, 2)	6.63	8.04	0.02	0.13	1.08	137.77	66.91	68.89
(8, 1.6, 4)	1.62	3.10	0.00	0.09	1.05	170.36	60.81	63.69
(8, 1.6, 8)	0.69	2.17	0.02	0.11	1.06	470.28	55.16	57.58

Table 3: CPU seconds required by the benchmark methods to compute one set of bid prices for the first airline network. The runtimes that are less than 5 milliseconds are indicated as zero.

$\operatorname{Problem}$			Upt	bound	obtained	l by					% gap be	etween L	<u>RDC ar</u>	pt	
(N, θ, κ)	LRD	LRDC	DLP	RLP	DPD	DRA	LAD	LADC	LRD	DLP	RLP	DPD	DRA	LAD	LADC
(4, 1.0, 2)	12,486	12,313	12,618	12,131	12,386	12,063	12,486	12,309	1.40	2.47	-1.48	0.60	-2.03	1.40	-0.04
(4, 1.0, 4)	17,661	17,442	17,815	17,327	17,528	17,165	17,661	17,440	1.26	2.14	-0.66	0.50	-1.59	1.26	-0.01
(4, 1.0, 8)	28,053	27,786	28,209	27,722	27,873	27,456	28,053	27,786	0.96	1.52	-0.23	0.31	-1.19	0.96	0.00
(4, 1.2, 2)	11,134	11,001	11,272	11,123	11,102	10,794	11,134	10,996	1.21	2.46	1.11	0.92	-1.88	1.21	-0.04
(4, 1.2, 4)	16,307	16,108	16,469	16,318	16,228	15,785	16,307	16,105	1.23	2.24	1.30	0.74	-2.01	1.23	-0.02
(4, 1.2, 8)	26,698	$26,\!439$	26,863	26,713	26,564	25,998	26,698	26,439	0.98	1.61	1.04	0.47	-1.67	0.98	0.00
(4, 1.6, 2)	9,308	9,207	9,451	9,321	9,299	8,966	9,308	9,197	1.10	2.66	1.24	1.00	-2.62	1.10	-0.11
(4, 1.6, 4)	14,472	14,309	14,649	14,510	14,421	13,902	14,472	14,304	1.14	2.38	1.41	0.79	-2.84	1.14	-0.03
(4, 1.6, 8)	24,862	24,637	25,043	24,905	24,754	24,056	24,862	24,636	0.91	1.65	1.09	0.47	-2.36	0.91	-0.01
(6, 1.0, 2)	14,675	14,556	14,818	14,205	14,671	14,065	14,675	14,553	0.82	1.81	-2.41	0.79	-3.37	0.82	-0.01
(6, 1.0, 4)	20,931	20,780	21,086	20,463	20,901	20,133	20,931	20,778	0.73	1.47	-1.52	0.58	-3.11	0.73	-0.01
(6, 1.0, 8)	33,466	33,284	33,621	33,004	33,404	32,475	33,466	33,281	0.55	1.01	-0.84	0.36	-2.43	0.55	-0.01
(6, 1.2, 2)	13,488	13,393	13,701	13,219	13,563	12,877	13,488	13,390	0.71	2.30	-1.30	1.26	-3.86	0.71	-0.02
(6, 1.2, 4)	19,658	19,530	19,932	19,444	19,759	18,752	19,658	19,526	0.66	2.06	-0.44	1.17	-3.98	0.66	-0.02
(6, 1.2, 8)	32,168	31,999	32,461	31,977	32,258	30,974	32,168	31,996	0.53	1.45	-0.07	0.81	-3.20	0.53	-0.01
(6, 1.6, 2)	11,274	11,175	11,550	11,263	11,415	10,703	11,274	11,168	0.89	3.35	0.79	2.15	-4.22	0.89	-0.06
(6, 1.6, 4)	17,316	17, 176	17,647	17,410	17,470	16,465	17, 316	17,166	0.82	2.75	1.36	1.72	-4.14	0.82	-0.06
(6, 1.6, 8)	29,790	29,597	30,153	29,919	29,922	28,554	29,790	29,590	0.65	1.88	1.09	1.10	-3.52	0.65	-0.02
(8, 1.0, 2)	15,904	15,780	16,099	15,307	15,936	15,056	15,904	15,773	0.79	2.03	-2.99	0.99	-4.58	0.79	-0.04
(8, 1.0, 4)	22,610	22,443	22,828	22,023	22,615	21,490	22,610	22,439	0.75	1.72	-1.87	0.77	-4.25	0.75	-0.02
(8, 1.0, 8)	36,063	35,863	36,286	35,489	36,031	34,696	36,063	35,858	0.56	1.18	-1.04	0.47	-3.25	0.56	-0.01
(8, 1.2, 2)	14,589	$14,\!480$	14,888	14,266	14,728	13,820	14,589	14,474	0.76	2.82	-1.47	1.72	-4.56	0.76	-0.03
(8, 1.2, 4)	21,238	21,077	21,588	20,957	21,378	20,100	21,238	21,073	0.77	2.43	-0.57	1.43	-4.64	0.77	-0.02
(8, 1.2, 8)	34,677	34,471	35,046	34,417	34,783	33,162	34,677	34,466	0.60	1.67	-0.15	0.91	-3.80	0.60	-0.01
(8, 1.6, 2)	12,153	12,077	12,521	12, 127	12,400	11,474	12,153	12,066	0.64	3.68	0.42	2.68	-4.99	0.64	-0.09
(8, 1.6, 4)	18,667	18,521	19, 139	18,757	18,941	17,531	18,667	18,512	0.79	3.34	1.27	2.27	-5.34	0.79	-0.05
(8, 1.6, 8)	32,070	31,866	32,589	32,197	32, 324	30,468	32,070	31,861	0.64	2.27	1.04	1.44	-4.39	0.64	-0.01
Average									0.85	2.16	-0.14	1.05	-3.33	0.85	-0.03

Table 4: Upper bounds on the optimal total expected revenues provided by the benchmark methods for the first airline network.

Problem		Tot	al expecté	evenue	e obtainec	d by			30	;ap betwee	en LRDC i	nd	
$(N, heta,\kappa)$	LRD	LRDC	DLP	RLP	DPD	LAD	LADC	LRD	DLP	RLP	DPD	LAD	LADC
(6, 1.0, 2)	23,560	23,748	23,467	23,475	23,726	23,362	23,754	0.79 🗸	1.18 √	$1.15 \checkmark$	$0.09 \odot$	1.62 🗸	-0.03 ⊙
(6, 1.0, 4)	33,126	33,650	32,881	33,076	33,592	33,046	33,645	$1.56 \checkmark$	2.29 \checkmark	$1.71 \checkmark$	0.17 \checkmark	$1.80 \checkmark$	$0.02 \odot$
(6, 1.0, 8)	52,273	53,789	51,693	52,661	53,721	52,472	53,786	$2.82 \checkmark$	$3.90 \checkmark$	$2.10 \checkmark$	0.13 \checkmark	$2.45 \checkmark$	$0.01 \odot$
(6, 1.2, 2)	21,470	21,930	21,457	21,577	21,891	21,535	21,918	$2.10 \checkmark$	$2.16 \checkmark$	$1.61 \checkmark$	0.18 \checkmark	$1.80 \checkmark$	$0.06 \odot$
(6, 1.2, 4)	30,658	31,769	30,475	31,132	31,751	31,031	31,770	$3.50 \checkmark$	$4.08 \checkmark$	$2.01 \checkmark$	0.06 \odot	$2.33 \checkmark$	0.00 \odot
(6, 1.2, 8)	49,033	51,931	48,484	50,903	51,939	50,139	51,935	$5.58 \checkmark$	$6.64 \checkmark$	$1.98 \checkmark$	-0.02 \odot	$3.45 \checkmark$	-0.01 ①
(6, 1.6, 2)	18,557	19,041	18,599	18,803	19,003	18,661	19,023	$2.54 \checkmark$	2.32 🗸	$1.25 \checkmark$	0.20 \checkmark	$1.99 \checkmark$	$0.10 \odot$
(6, 1.6, 4)	27,868	28,811	27,689	28,466	28,795	27,974	28,820	$3.28 \checkmark$	$3.89 \checkmark$	$1.20 \checkmark$	$0.06 \odot$	$2.91 \checkmark$	-0.03 \odot
(6, 1.6, 8)	46,411	48,867	45,892	48,071	48,814	46,648	48,869	$5.03 \checkmark$	~ 60.9	$1.63 \checkmark$	$0.11 \checkmark$	$4.54 \checkmark$	0.00 \odot
(10, 1.0, 2)	19,534	19,770	19,416	19,345	19,754	19,349	19,766	$1.19 \checkmark$	$1.79 \checkmark$	$2.15 \checkmark$	0.08 \odot	$2.13 \checkmark$	$0.02 \odot$
(10, 1.0, 4)	27,738	28,556	27,408	27,443	28,543	27,685	28,580	$2.86 \checkmark$	4.02 \checkmark	$3.90 \checkmark$	0.05 \odot	$3.05 \checkmark$	-0.08 ⊙
(10, 1.0, 8)	44,175	46,707	43,379	44,590	46,619	44,536	46,712	$5.42 \checkmark$	7.12 \checkmark	$4.53 \checkmark$	$0.19 \checkmark$	$4.65 \checkmark$	-0.01 \odot
(10, 1.2, 2)	17,357	17,655	17,240	17,211	17,621	17,039	17,655	$1.68 \checkmark$	2.35 🗸	$2.51 \checkmark$	$0.19 \odot$	$3.49 \checkmark$	$0.00 \odot$
(10, 1.2, 4)	25,097	26,340	24,549	25,101	26,307	25, 251	26,357	$4.72 \checkmark$	$6.80 \checkmark$	$4.70 \checkmark$	0.13 \odot	$4.13 \checkmark$	-0.06 \odot
(10, 1.2, 8)	40,922	44,412	39,270	41,472	44,355	41,234	44,424	7.86 🗸	$11.58 \checkmark$	$6.62 \checkmark$	0.13 \odot	$7.16 \checkmark$	-0.03 \odot
(10, 1.6, 2)	14,197	14,563	14,097	14,151	14,519	14,075	14,548	$2.51 \checkmark$	$3.20 \checkmark$	$2.83 \checkmark$	$0.30 \checkmark$	3.35 ✓	$0.10 \odot$
(10, 1.6, 4)	21,689	23,135	21,100	22,043	23,114	21,925	23, 129	$6.25 \checkmark$	8.80 🗸	$4.72 \checkmark$	0.09 \odot	$5.23 \checkmark$	$0.02 \odot$
(10, 1.6, 8)	37,179	41,203	35,289	38,823	41,150	37, 343	41,165	$9.77 \checkmark$	$14.35 \checkmark$	$5.78 \checkmark$	0.13 \odot	$9.37 \checkmark$	$0.09 \odot$
(14, 1.0, 2)	24,499	24,800	24,409	24,374	24,580	24, 325	24,774	$1.21 \checkmark$	$1.57 \checkmark$	$1.72 \checkmark$	$0.88 \checkmark$	$1.91 \checkmark$	$0.10 \odot$
(14, 1.0, 4)	34,689	35,725	34,281	34,531	35, 399	34,787	35,685	$2.90 \checkmark$	$4.04 \checkmark$	$3.34 \checkmark$	$0.91 \checkmark$	$2.63 \checkmark$	$0.11 \odot$
(14, 1.0, 8)	55,225	58,205	53,947	55,505	57,900	55, 399	58,161	$5.12 \checkmark$	7.32 🗸	$4.64 \checkmark$	0.52 \checkmark	$4.82 \checkmark$	$0.08 \odot$
(14, 1.2, 2)	22,102	22,496	21,989	21,991	22,368	21,973	22,457	$1.75 \checkmark$	$2.25 \checkmark$	$2.24 \checkmark$	$0.57 \checkmark$	$2.32 \checkmark$	$0.17 \checkmark$
(14, 1.2, 4)	31,700	33, 324	31,191	31,682	33, 135	31,679	33,250	$4.87 \checkmark$	$6.40 \checkmark$	$4.93 \checkmark$	$0.57 \checkmark$	$4.93 \checkmark$	$0.22\odot$
(14, 1.2, 8)	51,520	55,540	49,786	52,540	55,507	52,189	55,559	$7.24 \checkmark$	10.36 \checkmark	$5.40 \checkmark$	0.06 \odot	$6.03 \checkmark$	-0.03 \odot
(14, 1.6, 2)	18,628	19,003	18,369	18,600	18,863	18,512	18,969	$1.97 \checkmark$	3.33 ✓	$2.12 \checkmark$	0.73 \checkmark	$2.58 \checkmark$	$0.18 \odot$
(14, 1.6, 4)	27,891	29,630	27,162	28,367	29,508	27,946	29,580	$5.87 \checkmark$	8.33 🗸	$4.26 \checkmark$	0.41 \checkmark	$5.68 \checkmark$	$0.17 \checkmark$
(14, 1.6, 8)	47,208	51,882	45,280	49,181	51,779	47,474	51,849	$9.01 \checkmark$	12.72 V	5.21 \checkmark	0.20 \odot	8.50 🗸	$0.06 \checkmark$
Average								4.05	5.51	3.19	0.26	3.88	0.05

Table 5: Total expected revenues obtained by the benchmark methods for the second airline network when we recompute the bid prices five times over the planning horizon.

Problem		Tota	al expecte	d revenue	e obtained	l by			~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~	gap betwee	en LRDC i	and	
$(N, heta, \kappa)$	LRD	LRDC	DLP	RLP	DPD	LAD	LADC	LRD	DLP	RLP	DPD	LAD	LADC
(6, 1.0, 2)	23,697	23,729	23,646	23,657	23,688	23,545	23,725	0.13 \odot	0.35 🗸	$0.31 \checkmark$	$0.17 \checkmark$	$1.77 \checkmark$	$0.02 \odot$
(6, 1.0, 4)	33,441	33,588	33,306	33,544	33,539	33,404	33,588	$0.44 \checkmark$	$0.84 \checkmark$	$0.13 \odot$	$0.15 \checkmark$	$0.55 \checkmark$	0.00 \odot
(6, 1.0, 8)	53,033	53,726	52,661	53, 513	53,644	53,010	53,724	$1.29 \checkmark$	$1.98 \checkmark$	0.40 \odot	$0.15 \checkmark$	$1.33 \checkmark$	0.00 \odot
(6, 1.2, 2)	21,839	21,923	21,760	21,845	21,876	21,734	21,906	$0.38 \checkmark$	$0.75 \checkmark$	$0.36 \checkmark$	$0.22 \checkmark$	$0.86 \checkmark$	0.08 \odot
(6, 1.2, 4)	31,428	31,716	31,257	31,638	31,664	31,469	31,708	$0.91 \checkmark$	$1.45 \checkmark$	$0.25 \odot$	$0.16 \checkmark$	$0.78 \checkmark$	$0.02 \odot$
(6, 1.2, 8)	50,782	51,847	50,239	$51,\!432$	51,761	50,791	51,848	$2.05 \checkmark$	$3.10 \checkmark$	$0.80 \checkmark$	$0.16 \checkmark$	$2.04 \checkmark$	0.00 ①
(6, 1.6, 2)	18,898	19,005	18,836	19,010	18,951	18,832	18,999	$0.56 \checkmark$	$0.89 \checkmark$	-0.02 \odot	$0.29 \checkmark$	$0.91 \checkmark$	$0.03 \odot$
(6, 1.6, 4)	28,400	28,791	28,267	28,816	28,727	28,450	28,796	$1.36 \checkmark$	$1.82 \checkmark$	-0.09 ⊙	$0.22 \checkmark$	$1.18 \checkmark$	-0.02 ⊙
(6, 1.6, 8)	47,589	48,774	47,132	48,651	48,784	47,559	48,782	$2.43 \checkmark$	$3.37 \checkmark$	$0.25 \odot$	-0.02 \odot	$2.49 \checkmark$	-0.02 \odot
(10, 1.0, 2)	19,641	19,743	19,611	19,583	19,755	19,476	19,745	$0.52 \checkmark$	$0.67 \checkmark$	$0.81 \checkmark$	-0.06 ⊙	$1.35 \checkmark$	-0.01 ①
(10, 1.0, 4)	28,152	28,525	$27,\!841$	28,113	28,518	28,025	28,537	$1.31 \checkmark$	$2.40 \checkmark$	$1.45 \checkmark$	$0.03 \odot$	$1.76 \checkmark$	-0.04 \odot
(10, 1.0, 8)	45,296	46,633	44,418	45,912	46,529	45,092	46,625	$2.87 \checkmark$	$4.75 \checkmark$	$1.55 \checkmark$	$0.22 \checkmark$	3.30 \checkmark	$0.02 \odot$
(10, 1.2, 2)	17,576	17,655	17,560	17,551	17,605	17, 372	17,631	$0.45 \checkmark$	$0.54 \checkmark$	$0.59 \checkmark$	$0.28 \odot$	$1.60 \checkmark$	$0.14 \odot$
(10, 1.2, 4)	25,649	26,294	$25,\!459$	25,759	26,219	25,570	26,262	$2.45 \checkmark$	$3.18 \checkmark$	$2.04 \checkmark$	$0.29 \checkmark$	2.75 V	$0.12 \checkmark$
(10, 1.2, 8)	42,094	44,338	41, 312	$43,\!241$	$44,\!224$	$42,\!252$	44,333	$5.06 \checkmark$	$6.82 \checkmark$	$2.47 \checkmark$	$0.26 \checkmark$	$4.70 \checkmark$	$0.01 \odot$
(10, 1.6, 2)	14,498	14,541	14,416	14,464	14,485	14,307	14,554	$0.29~\odot$	$0.86 \checkmark$	$0.52 \checkmark$	$0.38 \checkmark$	$1.60 \checkmark$	-0.10 \odot
(10, 1.6, 4)	22,442	23,125	21,919	22,628	23,074	22,265	23,101	$2.96 \checkmark$	$5.22 \checkmark$	$2.15 \checkmark$	$0.22 \odot$	$3.72 \checkmark$	$0.11 \odot$
(10, 1.6, 8)	38,571	41,119	37,053	39,956	41,022	38,595	41,118	$6.20 \checkmark$	$9.89 \checkmark$	$2.83 \checkmark$	$0.24 \checkmark$	$6.14 \checkmark$	0.00 \odot
(14, 1.0, 2)	24,718	24,785	24,683	24,625	24,566	24,423	24,762	$0.27~\odot$	0.41 \odot	$0.65 \checkmark$	$0.88 \checkmark$	$1.46 \checkmark$	$0.10 \checkmark$
(14, 1.0, 4)	35,312	35,637	34,853	35,120	35, 333	34,929	35,602	$0.91 \checkmark$	$2.20 \checkmark$	$1.45 \checkmark$	$0.85 \checkmark$	$1.99 \checkmark$	$0.10 \checkmark$
(14, 1.0, 8)	56,518	58,084	55,294	56,832	57, 879	55,849	58,058	$2.70 \checkmark$	$4.80 \checkmark$	$2.16 \checkmark$	$0.35 \checkmark$	$3.85 \checkmark$	$0.04 \odot$
(14, 1.2, 2)	22,409	22,487	22,284	22, 338	22,310	22,167	22,465	$0.34 \odot$	$0.90 \checkmark$	$0.66 \checkmark$	$0.79 \checkmark$	$1.42 \checkmark$	$0.09 \odot$
(14, 1.2, 4)	32,545	33,154	31,996	32,509	32,983	32,403	33, 130	$1.84 \checkmark$	$3.49 \checkmark$	$1.95 \checkmark$	$0.52 \checkmark$	$2.26 \checkmark$	$0.07 \odot$
(14, 1.2, 8)	53,132	55,562	51, 279	53,829	55,472	$52,\!455$	55,546	$4.37 \checkmark$	7.71 🗸	$3.12 \checkmark$	$0.16 \odot$	$5.59 \checkmark$	$0.03 \odot$
(14, 1.6, 2)	18,894	18,979	18,757	18,843	18,776	18,733	18,929	0.45 \odot	$1.17 \checkmark$	$0.71 \checkmark$	$1.07 \checkmark$	$1.29 \checkmark$	$0.26 \checkmark$
(14, 1.6, 4)	28,612	29,572	28,056	28,859	29,431	28,411	29,542	$3.25 \checkmark$	$5.13 \checkmark$	$2.41 \checkmark$	$0.48 \checkmark$	$3.93 \checkmark$	$0.10 \odot$
(14, 1.6, 8)	48,423	51,829	46,829	49,996	51,776	48,210	51,810	$6.57 \checkmark$	$9.65 \checkmark$	$3.54 \checkmark$	$0.10 \odot$	$6.98 \checkmark$	$0.04 \odot$
Average								1.94	3.12	1.24	0.32	2.47	0.04

Table 6: Total expected revenues obtained by the benchmark methods for the second airline network when we recompute the bid prices 50 times over the planning horizon.

Problem			CPU	J second	ls for		
$(N, heta, \kappa)$	LRD	LRDC	DLP	RLP	DPD	LAD	LADC
(6, 1.0, 2)	0.78	2.51	0.00	0.05	1.38	2.39	3.80
(6, 1.0, 4)	0.75	2.21	0.00	0.05	1.38	2.74	4.08
(6, 1.0, 8)	0.75	2.43	0.00	0.05	1.38	2.52	3.80
(6, 1.2, 2)	0.77	2.29	0.00	0.05	1.16	6.59	7.72
(6, 1.2, 4)	0.75	1.96	0.00	0.05	1.16	6.78	7.60
(6, 1.2, 8)	0.84	1.68	0.00	0.05	1.16	6.89	7.73
(6, 1.6, 2)	0.75	1.66	0.00	0.05	0.88	7.00	7.81
(6, 1.6, 4)	0.75	1.67	0.00	0.05	0.88	5.77	6.61
(6, 1.6, 8)	0.75	1.66	0.00	0.05	0.88	5.69	6.69
(10, 1.0, 2)	5.31	8.87	0.00	0.05	3.27	12.25	15.41
(10, 1.0, 4)	5.09	8.56	0.02	0.05	3.31	14.89	17.86
(10, 1.0, 8)	4.98	8.65	0.02	0.05	3.31	14.73	17.77
(10, 1.2, 2)	5.56	8.62	0.00	0.05	2.75	23.66	26.91
(10, 1.2, 4)	5.14	8.14	0.02	0.05	2.75	22.27	23.98
(10, 1.2, 8)	5.02	8.54	0.02	0.06	2.78	22.45	24.86
(10, 1.6, 2)	5.74	8.05	0.02	0.05	2.11	21.64	24.02
(10, 1.6, 4)	5.23	7.57	0.00	0.06	2.13	22.47	26.63
(10, 1.6, 8)	5.03	7.88	0.00	0.06	2.09	22.22	25.00
(14, 1.0, 2)	4.65	11.92	0.03	0.06	5.92	42.13	51.16
(14, 1.0, 4)	3.29	10.23	0.03	0.06	5.94	38.64	45.77
(14, 1.0, 8)	3.16	9.94	0.03	0.06	5.94	42.16	47.41
(14, 1.2, 2)	4.14	9.65	0.02	0.08	4.98	51.61	56.08
(14, 1.2, 4)	3.27	9.34	0.02	0.08	4.99	44.20	52.17
(14, 1.2, 8)	3.18	9.02	0.02	0.08	5.05	48.38	57.63
(14, 1.6, 2)	4.10	8.96	0.02	0.08	3.80	54.23	62.52
(14, 1.6, 4)	3.31	8.10	0.02	0.08	3.81	55.13	61.38
(14, 1.6, 8)	3.16	7.86	0.03	0.08	3.80	53.06	58.63

Table 7: CPU seconds required by the benchmark methods to compute one set of bid prices for the second airline network. The runtimes that are less than 5 milliseconds are indicated as zero.

Proble	m		Upper b	ound obt	ained by				28 83	ap betwe	en LRD	C and	
(N, θ, N)	k) LRD	LRDC	DLP	RLP	DPD	LAD	LADC	LRD	DLP	RLP	DPD	LAD	LADC
(6, 1.0,	2) 24,77;	3 24,467	24,893	24,264	24,536	24,773	24,462	1.25	1.74	-0.83	0.28	1.25	-0.02
(6, 1.0,	(4) 35,16	5 34,796	35,299	34,650	34,857	35,165	34,789	1.06	1.45	-0.42	0.17	1.06	-0.02
(6, 1.0,	8) 55,97	7 55,530	56,112	55,429	55, 595	55,977	55, 529	0.81	1.05	-0.18	0.12	0.81	0.00
(6, 1.2,	2) 22,69	1 22,502	22,790	22,633	22,557	22,691	22,497	0.84	1.28	0.58	0.24	0.84	-0.02
(6, 1.2,	(4) 33,09'	7 32,821	33,196	33,013	32,878	33,097	32,820	0.84	1.14	0.58	0.17	0.84	0.00
(6, 1.2,	8) 53,90	9 $53,568$	54,008	53, 793	53,625	53,909	53,568	0.64	0.82	0.42	0.11	0.64	0.00
(6, 1.6,	2) 19,740	0 19,566	19,810	19,670	19,599	19,740	19,559	0.89	1.25	0.53	0.17	0.89	-0.03
(6, 1.6,	(4) 30,09	8 29,829	30,162	30,016	29,856	30,098	29,827	0.90	1.12	0.63	0.09	06.0	-0.01
(6, 1.6,	8) 50,91	1 50,561	50,974	50,790	50,589	50,911	50,561	0.69	0.82	0.45	0.06	0.69	0.00
(10, 1.0	,2) 20,98 ⁴	4 20,685	21,262	20,336	20,871	20,984	20,676	1.45	2.79	-1.69	0.90	1.45	-0.04
(10, 1.0)	, 4) 29, 89!	9 29,514	30, 229	29,303	29,739	29,899	29,508	1.31	2.42	-0.71	0.76	1.31	-0.02
(10, 1.0)	,8) 47,820	0 47,348	48,163	47,241	47,583	47,820	47, 343	1.00	1.72	-0.23	0.50	1.00	-0.01
(10, 1.2)	,2) 18,64!	5 18,462	18,923	18,638	18,669	18,645	18,450	0.99	2.50	0.96	1.12	0.99	-0.06
(10, 1.2)	$,4) \mid 27,56$	2 27,254	27,890	27,606	27,500	27,562	27,248	1.13	2.34	1.29	0.91	1.13	-0.02
(10, 1.2)	$, 8) 45, 48^{\circ}$	4 45,066	45,824	45,543	45,325	45,484	45,060	0.93	1.68	1.06	0.57	0.93	-0.01
(10, 1.6)	,2) 15,43	9 15,266	15,715	15,592	15,469	15,439	15,255	1.13	2.94	2.14	1.33	1.13	-0.07
(10, 1.6)	$,4) \mid 24,350$	0 24,047	24,680	24,558	24,295	24,350	24,041	1.26	2.63	2.12	1.03	1.26	-0.02
(10, 1.6)	, 8) 42, 27	4 41,858	42,613	42,494	42,118	42,274	41,855	0.99	1.80	1.52	0.62	0.99	-0.01
(14, 1.0	, 2) 26,68	8 26,454	27,077	25,809	26,717	26,688	26,452	0.88	2.35	-2.44	0.99	0.88	-0.01
(14, 1.0)	,4) 37,96	8 37,636	38,416	37,155	37,957	37,968	37,635	0.88	2.07	-1.28	0.85	0.88	0.00
(14, 1.0)	, 8) 60, 62	1 60,182	61,093	59,861	60,541	60,621	60,180	0.73	1.51	-0.53	0.60	0.73	0.00
(14, 1.2)	,2) 24,18	1 24,019	24,514	24,038	24,280	24,181	24,017	0.67	2.06	0.08	1.09	0.67	-0.01
(14, 1.2)	, 4) 35, 420	0 $35,121$	35,831	35,371	35,520	35,420	35,117	0.85	2.02	0.71	1.13	0.85	-0.01
(14, 1.2)	(8) 58,06	5 57,666	58,508	58,075	58,124	58,065	57,663	0.69	1.46	0.71	0.79	0.69	-0.01
(14, 1.6)	, 2) 20, 590	5 20,455	20,859	20,617	20,679	20,596	20,452	0.69	1.98	0.79	1.10	0.69	-0.01
(14, 1.6)	,4) 31,84'	7 31,626	32,171	31,929	31,917	31,847	31,618	0.70	1.73	0.96	0.92	0.70	-0.02
(14, 1.6	(8) 54,47	5 54,174	54,849	54,622	54,530	54,475	54,171	0.56	1.25	0.83	0.66	0.56	-0.01
Averag	e							0.92	1.77	0.30	0.64	0.92	-0.02

Table 8: Upper bounds on the optimal total expected revenues provided by the benchmark methods for the second airline network.