Price Competition under Linear Demand and Finite Inventories: Contraction and Approximate Equilibria

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Abstract

We consider a competitive pricing problem where there are multiple firms with limited inventories of substitutable products. Each firm chooses the prices that it charges for its product over a finite selling horizon. The demand of each firm jointly depends on the prices of all firms in a deterministic fashion through a linear demand function. The goal of each firm is to choose its prices to maximize its total revenue. We study two types of equilibrium. In equilibrium without recourse, each firm chooses its entire price trajectory at the beginning of the selling horizon. In equilibrium with recourse, each firm adjusts its price at each time period as a function of the current inventories of all of the firms. Although the demand of each firm is a deterministic function of the prices so that there is no uncertainty in the responses of the firms, there is a stark difference between equilibria with and without recourse. Considering the commonly studied diagonally dominant regime, where the demand of a firm is affected more by its price than the prices of the other firms, we show that the equilibrium without recourse exists and it is unique. In contrast, we demonstrate that the equilibrium with recourse may not exist or may not be unique. Motivated by this result, we look for approximate equilibrium with recourse. Considering a low influence regime, where the effect of the price of each firm on the demand of the others is diminishing, we show that the equilibria without recourse is an approximate equilibrium even when we allow the firms not to commit to a price trajectory at the beginning of the selling horizon.

1 Introduction

In many practical situations, multiple firms selling substitutable products set their prices competitively to sell limited inventories over a finite selling horizon, when the demand of each firm jointly depends on the prices charged by all firms. For example, airlines competitively set the prices for their limited seat inventories in a particular market. Firms selling electronic products take the prices of their competitors into consideration when setting their prices. In this paper, we consider multiple firms with limited inventories of substitutable products. Each firm chooses the prices that it charges for its product over a finite selling horizon. The demand that each firm faces is a deterministic function of the prices charged by all of the firms, where the demand of a firm is linearly decreasing in its price and linearly increasing in the prices of the other firms. Each firm chooses its prices over a finite selling horizon to maximize its total revenue.

Main Contributions. We study two types of equilibrium for the competitive pricing setting described in the previous paragraph. In equilibrium without recourse, at the beginning of the selling horizon, each firm chooses the prices that it charges over the whole selling horizon and commits to this price trajectory, under the assumption that the other firms do the same. In equilibrium

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with recourse, at each time period in the selling horizon, each firm observes the inventories of all of the firms and chooses its price at the current time period, again under the assumption that the other firms do the same. Despite the fact that the demand of each firm is a deterministic function of the prices so that there is no uncertainty in the responses of the firms, we show a clear contrast between the equilibrium without recourse and the equilibrium with recourse.

We consider the diagonal dominant regime, where the price charged by each firm affects its demand more than the prices charged by the other firms. In other words, if all of the competitors of a firm decrease their prices by a certain amount, then the firm can decrease its price by the same amount to ensure that its demand does not decrease. This regime is rather standard in the existing literature and it is used in, for example, Allon and Federgruen (2007) and Gallego and Hu (2014). Focusing on the equilibrium without recourse, we show that the best response of each firm to the price trajectories of the other firms is a contraction mapping, when viewed as a function of the prices of the other firms. In this case, it immediately follows that the equilibrium without recourse always exists and it is unique. To our knowledge, this contraction property in the setting of price competition with limited inventories is new.

We give counterexamples to show that the equilibrium with recourse may not exist or may not be unique. Motivated by this observation, we look for an approximate equilibrium that is guaranteed to exist. In particular, we consider the setting where a firm can adjust its price at each time period based on the current inventories of all of the firms. We call a strategy profile for the firms an ϵ-equilibrium with recourse if no firm can improve its total revenue by more than ϵ by deviating from its strategy profile. We consider a low influence regime, where the effect of the price of a firm on the demand of another firm is diminishing, which naturally holds when the number of firms is large. We show that the equilibrium without recourse, which we know to uniquely exist, is an ϵ-equilibrium with recourse even when the firms may adjust their prices. So, intuitively speaking, an ϵ-equilibrium with recourse is expected to exist when the number of firms is large.

Our results fills a gap in a fundamental class of revenue management problems. Although there is no uncertainty in the responses of the firms, the equilibria with and without recourse are not the same concept and they can actually be qualitatively rather different. While the equilibrium without recourse uniquely exists, one cannot say the same thing for the equilibrium with recourse. Also, as far as we are aware, our contraction argument for showing the existence and uniqueness of the equilibrium without recourse does not appear in the literature. This argument becomes surprisingly effective when dealing with linear demand functions, but it is still an open question whether similar arguments hold for other demand functions. Lastly, our results indicate that the equilibrium without recourse in a low influence regime turns out to be an approximate equilibrium with recourse, even when the firms are allowed to not to commit to a price trajectory.

Literature Review. Similar to us, Gallego and Hu (2014) consider price competition among multiple firms with limited inventories over a finite selling horizon. In the diagonally dominant regime, they show the unique existence of the equilibrium without recourse, but their argument does not show the contraction of the best response. Having the contraction property allows finding the equilibrium without recourse by successively computing the best response of each firm to the others. Furthermore, for the setting we consider, we show that the equilibrium with recourse can be different from the equilibrium withour recourse and the former equilibrium may not exist or may not be unique, but the equilibrium without recourse may be a good approximation to the equilibrium with recourse in the low influence regime.

There are a number of papers that study price competition over a single period. Milgrom and Roberts (1990) show that pure Nash equilibrium exists for a wide class of so called supermodular demand models. Gallego et al. (2006) provide sufficient conditions for uniqueness of equilibrium in the Bertrand game when the demands of the firms are nonlinear functions of the prices, there is a cost associated with satisfying a certain volume of demand and each firm is interested in maximizing its expected profit. In particular, the authors use a slightly more general version of the multinomial demand model to capture the relationship between price and demand. The cost is a nonlinear function of the demand volume. Pierson et al. (2013) identify the conditions for existence and uniqueness of pure Nash equilibrium when the demands are characterized by a mixture of multinomial logit models and the cost of satisfying a certain volume of demand is linear in the demand volume. Gallego and Wang (2014) consider the price competition between multiple firms when the relationship between demand and price is characterized by the nested logit model and provide conditions to ensure the existence and uniqueness of the equilibrium. Nazerzadeh and Perakis (2015) prove the existence of pure strategy equilibrium in a price competition game between two suppliers when capacity is private information.

Considering the papers on price competition over multiple time periods, Levin et al. (2009) study a stochastic game when there are strategic consumers choosing the time to purchase. Lin and Sibdari (2009) study a competitive pricing problem when the relationship between demand and price is captured by the multinomial logit model and inventory levels are public information. Martinez-de Albeniz and Talluri (2011) study the pricing game between two firms with limited inventories facing stochastic demand. The authors characterize the unique subgame perfect equilibrium. Liu and Zhang (2013) show that there exists a unique pure strategy Markov perfect equilibrium in a pricing game between two firms offering vertically differentiated products.

Organization. In Section 2, we describe the competitive pricing setting, formulate the optimization problem that computes the best response of a firm when the firms commit to price trajectories at the beginning of the selling horizon and show that the best response is a contraction mapping when viewed as a function of the prices of the other firms, which allows us to conclude that the equilibrium without recourse uniquely exists. In Section 3, we define the equilibrium with recourse and provide counterexamples to show that the equilibrium with recourse may not exist or may not be unique. In Section 4, we show that the equilibrium without recourse is an ϵ-equilibrium with recourse in a low influence regime. In Section 5, we conclude.

2 Equilibrium without Recourse

There are n firms indexed by $N = \{1, \ldots, n\}$. Firm i has c_i units of initial inventory, which cannot be replenished over the selling horizon. There are τ time periods in the selling horizon indexed by $T = \{1, \ldots, \tau\}$. We use p_i^t to denote the price charged by firm i at time period t. Using $\boldsymbol{p}^t = (p_1^t, \ldots, p_n^t)$ to denote the prices charged all of the firms at time period t, the demand faced by firm i at time period t is given by $D_i^t(\boldsymbol{p}^t) = \alpha_i^t - \beta_i^t\, p_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t$, where $\alpha_i^t > 0$, $\beta_i^t > 0$ and $\gamma_{i,j}^t > 0$. We assume that the price charged by each firm affects its demand more than the prices charged by the other firms, in the sense that $\sum_{j \neq i} \gamma_{i,j}^t < \beta_i^t$ for all $i \in N$, $t \in T$. Also, using $\boldsymbol{p}_{-i}^t = (p_1^t, \ldots, p_{i-1}^t, p_{i+1}^t, \ldots, p_n^t)$ to denote the prices charged by firms other than firm i at time period t, to avoid negative demand quantities, we restrict the strategy space of the firms such that each firm i charges the price p_i^t at time period t that satisfies $\alpha_i^t - \beta_i^t\, p_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t \geq 0$, given the prices $\boldsymbol{p}_{-i}^t$ charged by the other firms. If the firms other than firm i commit to the price trajectories $\boldsymbol{p}_{-i} = \{\boldsymbol{p}_{-i}^t : t \in T\}$, then we can obtain the best response of firm i by solving the problem

$$\max \left\{ \sum_{t \in T} \Big(\alpha_i^t - \beta_i^t\, p_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t \Big)\, p_i^t \;:\; \sum_{t \in T} \Big(\alpha_i^t - \beta_i^t\, p_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t \Big) \leq c_i, \right.$$
$$\left. \alpha_i^t - \beta_i^t\, p_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t \geq 0 \;\; \forall\, t \in T, \quad p_i^t \geq 0 \;\; \forall\, t \in T \right\}. \tag{1}$$

Since $\beta_i^t > 0$, problem (1) has a strictly convex objective function and linear constraints, which implies that the best response of firm i is unique.

Using the non-negative dual multipliers v_i and $\{u_i^t : t \in T\}$ for the first and second constraint in problem (1), the Karush-Kuhn-Tucker (KKT) conditions for this problem are

$$\Big(\sum_{t \in T} \Big(\alpha_i^t - \beta_i^t\, p_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t \Big) - c_i \Big)\, v_i = 0, \qquad \Big(\alpha_i^t - \beta_i^t\, p_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t \Big)\, u_i^t = 0 \;\; \forall\, t \in T, \tag{2}$$
$$\alpha_i^t - 2\, \beta_i^t\, p_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t + \beta_i^t\, (v_i - u_i^t) = 0 \;\; \forall\, t \in T.$$

Since problem (1) has a concave objective function and linear constraints, the KKT conditions above are necessary and sufficient at optimality; see Boyd and Vandenberghe (2004). In other words, for a feasible solution $\{p_i^t : t \in T\}$ to problem (1), there exist corresponding non-negative dual multipliers v_i and $\{u_i^t : t \in T\}$ that satisfy the KKT conditions in (2) if and only if $\{p_i^t : t \in T\}$ is the optimal solution to problem (1). Note that we do not associate dual multipliers with the constraints $p_i^t \geq 0$ for all $t \in T$ in problem (1) since it is never optimal for firm i to charge a negative price. Therefore, we can actually view the constraints $p_i^t \geq 0$ for all $t \in T$ as redundant constraints. We use the KKT conditions in (2) extensively to characterize the best response of firm i to the price trajectories $\boldsymbol{p}_{-i}$ of the other firms. In the rest of this section, we exclusively focus on the *strategies without recourse*, where each firm i commits to a price trajectory $\{p_i^t : t \in T\}$ at the beginning of the selling horizon and does not adjust these prices during the course of the selling horizon. If the price trajectory $\{p_i^t : t \in T\}$ chosen by each firm i is the best response to the price trajectories $\boldsymbol{p}_{-i}$ chosen by the other firms, then we say that the price trajectories $\{\boldsymbol{p}^t : t \in T\}$ chosen by the firms is an *equilibrium without recourse*. We show that there exists a unique equilibrium without recourse. Furthermore, if we start with any price trajectory $\{\boldsymbol{p}^t : t \in T\}$ for the firms and successively compute the best response of each firm to the price trajectories of the other firms, then the best response of each firm forms a contraction mapping when viewed as a function of the prices charged by the other firms. Using this result, we show that there exists a unique equilibrium without recourse. To capture the best response of firm i to the prices charged by the other firms, we define the set of time periods

$$\mathcal{T}_i(\nu, \boldsymbol{p}_{-i}) = \left\{ t \in T : \frac{\alpha_i^t + \sum_{j \neq i} \gamma_{i,j}^t \, p_j^t}{\beta_i^t} > \nu \right\}.$$

In the next lemma, we use $\mathcal{T}_i(\nu, \boldsymbol{p}_{-i})$ to give a succinct characterization of the solution $\{p_i^t : t \in T\}$ and the corresponding dual multipliers v_i and $\{u_i^t : t \in T\}$ that satisfy the KKT conditions.

Lemma 1 *If a feasible solution $\{p_i^t : t \in T\}$ to problem* (1) *and the corresponding non-negative dual multipliers v_i and $\{u_i^t : t \in T\}$ satisfy the KKT conditions in* (2)*, then we have*

$$p_i^t = \begin{cases} \dfrac{\alpha_i^t + \sum_{j \neq i} \gamma_{i,j}^t \, p_j^t}{2\,\beta_i^t} + \dfrac{v_i}{2} & \text{if } t \in \mathcal{T}_i(v_i, \boldsymbol{p}_{-i}) \\ \dfrac{\alpha_i^t + \sum_{j \neq i} \gamma_{i,j}^t \, p_j^t}{\beta_i^t} & \text{if } t \notin \mathcal{T}_i(v_i, \boldsymbol{p}_{-i}), \end{cases} \qquad u_i^t = \begin{cases} 0 & \text{if } t \in \mathcal{T}_i(v_i, \boldsymbol{p}_{-i}) \\ v_i - \dfrac{\alpha_i^t + \sum_{j \neq i} \gamma_{i,j}^t \, p_j^t}{\beta_i^t} & \text{if } t \notin \mathcal{T}_i(v_i, \boldsymbol{p}_{-i}). \end{cases}$$

Proof. Since the solution $\{p_i^t : t \in T\}$, along with the dual multipliers v_i and $\{u_i^t : t \in T\}$, satisfies the KKT conditions in (2), solving for u_i^t in the third KKT condition, we have

$$u_i^t = \frac{\alpha_i^t + \sum_{j \neq i} \gamma_{i,j}^t \, p_j^t}{\beta_i^t} - 2\,p_i^t + v_i. \tag{3}$$

For notational brevity, we let $\Delta_i^t = (\alpha_i^t + \sum_{j \neq i} \gamma_{i,j}^t \, p_j^t)/\beta_i^t$. Therefore, we can write (3) as $u_i^t = \Delta_i^t - 2\,p_i^t + v_i$. Furthermore, noting that $\beta_i^t > 0$ and dividing the second KKT condition in (2) by β_i^t, we observe that $(\Delta_i^t - p_i^t)\,u_i^t = 0$ for all $t \in T$. Consider any $t \in \mathcal{T}_i(v_i, \boldsymbol{p}_{-i})$. By the definition of $\mathcal{T}_i(v_i, \boldsymbol{p}_{-i})$, we have $\Delta_i^t > v_i$. In this case, by (3), it follows that $u_i^t = \Delta_i^t - 2\,p_i^t + v_i < 2\,(\Delta_i^t - p_i^t)$. Multiplying the last chain of inequalities by u_i^t and noting that $(\Delta_i^t - p_i^t)\,u_i^t = 0$, we get $(u_i^t)^2 \leq 0$, which implies that $u_i^t = 0$. Using this value of u_i^t in (3) and solving for p_i^t, we have $p_i^t = \Delta_i^t/2 + v_i/2$. Therefore, the desired result holds for any $t \in \mathcal{T}_i(v_i, \boldsymbol{p}_{-i})$. Consider any $t \notin \mathcal{T}_i(v_i, \boldsymbol{p}_{-i})$. By the definition of $\mathcal{T}_i(v_i, \boldsymbol{p}_{-i})$, we have $\Delta_i^t \leq v_i$. In this case, using (3), it follows that $u_i^t = \Delta_i^t - 2\,p_i^t + v_i \geq 2\,(\Delta_i^t - p_i^t)$. Multiplying the last chain of inequalities by $\Delta_i^t - p_i^t$ and noting that $(\Delta_i^t - p_i^t)\,u_i^t = 0$, we have $(\Delta_i^t - p_i^t)^2 \leq 0$, which implies that $p_i^t = \Delta_i^t$. Using this value of p_i^t in (3) and noting the definition of Δ_i^t, we get $u_i^t = v_i - \Delta_i^t$. Therefore, the desired result holds for any $t \notin \mathcal{T}_i(v_i, \boldsymbol{p}_{-i})$. □

By Lemma 1, we can characterize the solution $\{p_i^t : t \in T\}$ and the dual multipliers v_i and $\{u_i^t : t \in T\}$ that satisfy the KKT conditions in (2) only by using the value of v_i. If we know the value of v_i, then we can compute the set of time periods $\mathcal{T}_i(v_i, \boldsymbol{p}_{-i})$, in which case, we can choose the values of $\{p_i^t : t \in T\}$ and $\{u_i^t : t \in \mathcal{T}_i\}$ as given in Lemma 1. Throughout the rest of this section, we indeed choose the values of $\{p_i^t : t \in T\}$ and $\{u_i^t : t \in \mathcal{T}_i\}$ as given in Lemma 1, since we are interested in solution thats satisfy the KKT conditions. Naturally, we do not know the value of v_i that allows us to obtain the solution $\{p_i^t : t \in T\}$ and the dual multipliers v_i and $\{u_i^t : t \in T\}$ that satisfy the KKT conditions. In the next lemma, we give a characterization of the value of v_i that corresponds to the solution $\{p_i^t : t \in T\}$ and the dual multipliers v_i and $\{u_i^t : t \in T\}$ satisfying the KKT conditions in (2). In particular, we consider the function

$$G_i(\nu, \boldsymbol{p}_{-i}) = \begin{cases} \displaystyle\sum_{t \in \mathcal{T}_i(\nu, \boldsymbol{p}_{-i})} \Big(\alpha_i^t - \beta_i^t\, \nu + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t\Big) - 2\, c_i & \text{if } \nu > 0 \\ \max\Big\{ \displaystyle\sum_{t \in \mathcal{T}_i(\nu, \boldsymbol{p}_{-i})} \Big(\alpha_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t\Big) - 2\, c_i, 0 \Big\} & \text{if } \nu = 0. \end{cases}$$

In the appendix, Lemma 7 shows that $G_i(\cdot, \boldsymbol{p}_{-i})$ is strictly decreasing over some $[0, \nu^*]$ and has a unique root. In the next lemma, we use its root to characterize a solution to the KKT conditions.

Lemma 2 *If a feasible solution $\{p_i^t : t \in T\}$ to problem* (1) *and the corresponding non-negative dual multipliers v_i and $\{u_i^t : t \in T\}$ satisfy the KKT conditions in* (2)*, then we have $G_i(v_i, \boldsymbol{p}_{-i}) = 0$.*

Proof. As in the proof of Lemma 1, we let $\Delta_i^t = (\alpha_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t)/\beta_i^t$ for notational brevity. By Lemma 1, we have $p_i^t = (\Delta_i^t + v_i)/2$ for all $t \in \mathcal{T}_i(v_i, \boldsymbol{p}_{-i})$ and $p_i^t = \Delta_i^t$ for all $t \notin \mathcal{T}_i(v_i, \boldsymbol{p}_{-i})$. First, we assume that $v_i = 0$. Since $\alpha_i^t > 0$, we have $\mathcal{T}_i(v_i, \boldsymbol{p}_{-i}) = T$ by the definition of $\mathcal{T}_i(v_i, \boldsymbol{p}_{-i})$, which implies that $p_i^t = (\Delta_i^t + v_i)/2 = \Delta_i^t/2$ for all $t \in T$. In this case, we obtain $\frac{1}{2} \sum_{t \in T} \beta_i^t\, \Delta_i^t = \sum_{t \in T} \beta_i^t (\Delta_i^t - p_i^t) = \sum_{t \in T} (\alpha_i^t - \beta_i^t\, p_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t) \leq c_i$, where the second equality uses the definition of Δ_i^t and the inequality follows from the fact that $\{p_i^t : t \in T\}$ is a feasible solution to problem (1). The last chain of inequalities imply that $\sum_{t \in T} \beta_i^t\, \Delta_i^t - 2\, c_i \leq 0$. Noting the definition of Δ_i^t and the fact that $\mathcal{T}_i(v_i, \boldsymbol{p}_{-i}) = T$, we obtain $\sum_{t \in \mathcal{T}_i(v_i, \boldsymbol{p}_{-i})} (\alpha_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t) - 2\, c_i \leq 0$, which implies that $G_i(v_i, \boldsymbol{p}_{-i}) = G_i(0, \boldsymbol{p}_{-i}) = 0$. Therefore, the desired result holds when $v_i = 0$. Second, we assume that $v_i > 0$. Using the fact that $p_i^t = (\Delta_i^t + v_i)/2$ for all $t \in \mathcal{T}_i(v_i, \boldsymbol{p}_{-i})$ and $p_i^t = \Delta_i^t$ for all $t \notin \mathcal{T}_i(v_i, \boldsymbol{p}_{-i})$, we have $\sum_{t \in T} (\alpha_i^t - \beta_i^t\, p_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t) = \sum_{t \in T} \beta_i^t\, (\Delta_i^t - p_i^t) = \sum_{t \in \mathcal{T}_i(v_i, \boldsymbol{p}_{-i})} \beta_i^t\, (\Delta_i^t - v_i)/2$. Since $v_i > 0$, by the first KKT condition in (2), we also have $c_i = \sum_{t \in T} (\alpha_i^t - \beta_i^t\, p_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t)$. In this case, by the last chain of equalities, we get

$$c_i = \frac{1}{2} \sum_{t \in \mathcal{T}_i(v_i, \boldsymbol{p}_{-i})} \beta_i^t\, (\Delta_i^t - v_i).$$

By the definition of Δ_i^t, the equality above is equivalent to $\sum_{t \in \mathcal{T}_i(v_i, \boldsymbol{p}_{-i})} (\alpha_i^t - \beta_i v_i + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t) - 2\, c_i = 0$, which implies that $G_i(v_i, \boldsymbol{p}_{-i}) = 0$. Therefore, the desired result holds when $v_i > 0$. $\square$

By Lemma 2, if a feasible solution $\{p_i^t : t \in T\}$ to problem (1) and the corresponding non-negative dual multipliers v_i and $\{u_i^t : t \in T\}$ satisfy the KKT conditions in (2), then v_i must be the unique root of $G_i(\cdot, \boldsymbol{p}_{-i})$. Also, by Lemma 1, the values of $\{p_i^t : t \in T\}$ and $\{u_i^t : t \in T\}$ must be given as in Lemma 1. In the next theorem, we use these results to show that the best response of firm i is a contraction mapping when viewed as a function of the prices of the other firms.

Theorem 3 *Let $\{p_i^t(\boldsymbol{p}_{-i}) : t \in T\}$ be the optimal solution to problem (1) as a function of the prices charged by the firms other than firm i. For any two price trajectories $\hat{\boldsymbol{p}}_{-i} = \{\hat{\boldsymbol{p}}_{-i}^t : t \in T\}$ and $\tilde{\boldsymbol{p}}_{-i} = \{\tilde{\boldsymbol{p}}_{-i}^t : t \in T\}$ adopted by the firms other than firm i, we have*

$$|p_i^t(\hat{\boldsymbol{p}}_{-i}) - p_i^t(\tilde{\boldsymbol{p}}_{-i})| \leq \max_{t \in T} \left\{ \frac{\sum_{j \neq i} \gamma_{i,j}^t \, |\hat{p}_j^t - \tilde{p}_j^t|}{\beta_i^t} \right\}.$$

Proof. For notational brevity, we let $\hat{p}_i^t = p_i^t(\hat{\boldsymbol{p}}_{-i})$ and $\tilde{p}_i^t = p_i^t(\tilde{\boldsymbol{p}}_{-i})$. In other words, $\{\hat{p}_i^t : t \in T\}$ is the optimal solution to problem (1) when we solve this problem after replacing $\boldsymbol{p}_{-i}$ with $\hat{\boldsymbol{p}}_{-i}$. Similarly, $\{\tilde{p}_i^t : t \in T\}$ is the optimal solution to problem (1) when we solve this problem after replacing $\boldsymbol{p}_{-i}$ with $\tilde{\boldsymbol{p}}_{-i}$. Also, we let $\hat{v}_i$ and $\tilde{v}_i$ be such that $G_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i}) = 0$ and $G_i(\tilde{v}_i, \tilde{\boldsymbol{p}}_{-i}) = 0$. Without loss of generality, we assume that $\hat{v}_i \geq \tilde{v}_i$. Otherwise, we interchange the roles of $\hat{v}_i$ and $\tilde{v}_i$. Finally, we let $\hat{\Delta}_i^t = (\alpha_i^t + \sum_{j \neq i} \gamma_{i,j}^t \, \hat{p}_j^t)/\beta_i^t$ and $\tilde{\Delta}_i^t = (\alpha_i^t + \sum_{j \neq i} \gamma_{i,j}^t \, \tilde{p}_j^t)/\beta_i^t$ for notational brevity. Note that $|\hat{\Delta}_i^t - \tilde{\Delta}_i^t| \leq \sum_{j \neq i} \gamma_{i,j}^t \, |\hat{p}_j^t - \tilde{p}_j^t|/\beta_i^t$. In this case, using $M_i = \max_{t \in T}\{\sum_{j \neq i} \gamma_{i,j}^t \, |\hat{p}_j^t - \tilde{p}_j^t|/\beta_i^t\}$, we have $|\hat{\Delta}_i^t - \tilde{\Delta}_i^t| \leq M_i$ for all $t \in T$. We proceed to examining four cases to show that $|\hat{p}_i^t - \tilde{p}_i^t| \leq \frac{1}{2} M_i + \frac{1}{2} \max\{M_i, \hat{v}_i - \tilde{v}_i\}$ for all $t \in T$. First, we assume that $t \in \mathcal{T}_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i})$ and $t \in \mathcal{T}_i(\tilde{v}_i, \tilde{\boldsymbol{p}}_{-i})$. Using Lemma 1, we have $|\hat{p}_i^t - \tilde{p}_i^t| = \frac{1}{2}|\hat{\Delta}_i^t + \hat{v}_i - \tilde{\Delta}_i^t - \tilde{v}_i^t| \leq \frac{1}{2}|\hat{\Delta}_i^t - \tilde{\Delta}_i^t| + \frac{1}{2}(\hat{v}_i - \tilde{v}_i) \leq \frac{1}{2} M_i + \frac{1}{2}(\hat{v}_i - \tilde{v}_i) \leq \frac{1}{2} M_i + \frac{1}{2} \max\{M_i, \hat{v}_i - \tilde{v}_i\}$, as desired. Second, we assume that $t \notin \mathcal{T}_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i})$ and $t \notin \mathcal{T}_i(\tilde{v}_i, \tilde{\boldsymbol{p}}_{-i})$. Using Lemma 1 once more, we have $|\hat{p}_i^t - \tilde{p}_i^t| = |\hat{\Delta}_i^t - \tilde{\Delta}_i^t| \leq M_i \leq \frac{1}{2} M_i + \frac{1}{2} \max\{M_i, \hat{v}_i - \tilde{v}_i\}$, as desired. Third, we assume that $t \in \mathcal{T}_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i})$ and $t \notin \mathcal{T}_i(\tilde{v}_i, \tilde{\boldsymbol{p}}_{-i})$. Since $t \in \mathcal{T}_i(v_i, \hat{\boldsymbol{p}}_{-i})$, we have $\hat{\Delta}_i^t > \hat{v}_i$, which implies $\hat{v}_i - \tilde{\Delta}_i^t < \hat{\Delta}_i^t - \tilde{\Delta}_i^t \leq M_i$. Also, since $t \notin \mathcal{T}_i(\tilde{v}_i, \tilde{\boldsymbol{p}}_{-i})$, we have $\tilde{\Delta}_i^t \leq \tilde{v}_i$, which implies $\hat{v}_i - \tilde{\Delta}_i^t \geq \hat{v}_i - \tilde{v}_i$. Noting the last two inequalities, it follows that $|\hat{v}_i - \tilde{\Delta}_i^t| \leq \max\{M_i, \hat{v}_i - \tilde{v}_i\}$. In this case, using Lemma 1 one last time and using the fact that $|\hat{v}_i - \tilde{\Delta}_i^t| \leq \max\{M_i, \hat{v}_i - \tilde{v}_i\}$, we obtain $|\hat{p}_i^t - \tilde{p}_i^t| = |\frac{1}{2}\hat{\Delta}_i^t + \frac{1}{2}\hat{v}_i - \tilde{\Delta}_i^t| \leq \frac{1}{2}\,|\hat{\Delta}_i^t - \tilde{\Delta}_i^t| + \frac{1}{2}\,|\hat{v}_i - \tilde{\Delta}_i^t| \leq \frac{1}{2}\, M_i + \frac{1}{2}\,|\hat{v}_i - \tilde{\Delta}_i^t| \leq \frac{1}{2}\, M_i + \frac{1}{2}\, \max\{M_i, \hat{v}_i - \tilde{v}_i\}$, as desired. Fourth, we assume that $t \notin \mathcal{T}_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i})$ and $t \in \mathcal{T}_i(\tilde{v}_i, \tilde{\boldsymbol{p}}_{-i})$, in which case, we can follow the same argument in the third case to obtain $|\hat{p}_i^t - \tilde{p}_i^t| \leq \frac{1}{2}\, M_i + \frac{1}{2}\, \max\{M_i, \hat{v}_i - \tilde{v}_i\}$. The preceding discussion shows that $|\hat{p}_i^t - \tilde{p}_i^t| \leq \frac{1}{2}\, M_i + \frac{1}{2}\, \max\{M_i, \hat{v}_i - \tilde{v}_i\}$. If $\hat{v}_i \leq M_i$, then noting that $\tilde{v}_i \geq 0$, the last inequality implies that $|\hat{p}_i^t - \tilde{p}_i^t| \leq M_i$, which is the result we want to show! In the rest of the proof, we proceed under the assumption that $\hat{v}_i > M_i$.

Consider the function $G_i(\cdot, \tilde{\boldsymbol{p}}_{-i})$. By Lemma 7 in the appendix, the function $G_i(\cdot, \tilde{\boldsymbol{p}}_{-i})$ is strictly decreasing over the interval $[0, \nu^*)$ for some ν^* and constant over the interval $[\nu^*, \infty)$. By the same lemma, we also have $G_i(\nu^*, \tilde{\boldsymbol{p}}_{-i}) = -2\, c_i < 0$. In the rest of the proof, we show that $G_i(\hat{v}_i - M_i, \tilde{\boldsymbol{p}}_{-i}) \geq 0$. Also, we have $G_i(\tilde{v}_i, \tilde{\boldsymbol{p}}_{-i}) = 0$ by Lemma 2. In this case, since $G_i(\nu^*, \tilde{\boldsymbol{p}}_{-i}) < 0$ and $G_i(\cdot, \tilde{\boldsymbol{p}}_{-i})$ is strictly decreasing over the interval $[0, \nu^*)$ and constant over the interval $[\nu^*, \infty)$, having $G_i(\hat{v}_i - M_i, \tilde{\boldsymbol{p}}_{-i}) \geq 0$ and $G_i(\tilde{v}_i, \tilde{\boldsymbol{p}}_{-i}) = 0$ implies that $\hat{v}_i - M_i \leq \tilde{v}_i$. Therefore, we have

$\hat{v}_i - \tilde{v}_i \leq M_i$, so that we get $|\hat{p}_i^t - \tilde{p}_i^t| \leq \frac{1}{2} M_i + \frac{1}{2} \max\{M_i, \hat{v}_i - \tilde{v}_i\} = M_i$, which is the result we want to show. It remains to show that $G_i(\hat{v}_i - M_i, \tilde{\boldsymbol{p}}_{-i}) \geq 0$. Using $\mathbf{1}(\cdot)$ to denote the indicator function, since $\hat{v}_i > M_i$, by the definition of $G_i(\cdot, \tilde{\boldsymbol{p}}_{-i})$, we have

$$\begin{aligned}
G_i(\hat{v}_i - M_i, \tilde{\boldsymbol{p}}_{-i}) &= \sum_{t \in \mathcal{T}_i(\hat{v}_i - M_i, \tilde{\boldsymbol{p}}_{-i})} \Big(\alpha_i^t - \beta_i^t\, (\hat{v}_i - M_i) + \sum_{j \neq i} \gamma_{i,j}^t\, \tilde{p}_j^t \Big) - 2\, c_i \\
&= \sum_{t \in \mathcal{T}_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i})} \Big(\alpha_i^t - \beta_i^t\, (\hat{v}_i - M_i) + \sum_{j \neq i} \gamma_{i,j}^t\, \tilde{p}_j^t \Big) - 2\, c_i \\
&\quad + \sum_{t \in T} \mathbf{1}(t \in \mathcal{T}_i(\hat{v}_i - M_i, \tilde{\boldsymbol{p}}_{-i}) \setminus \mathcal{T}_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i})) \Big(\alpha_i^t - \beta_i^t\, (\hat{v}_i - M_i) + \sum_{j \neq i} \gamma_{i,j}^t\, \tilde{p}_j^t \Big) \\
&\quad - \sum_{t \in T} \mathbf{1}(t \in \mathcal{T}_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i}) \setminus \mathcal{T}_i(\hat{v}_i - M_i, \tilde{\boldsymbol{p}}_{-i})) \Big(\alpha_i^t - \beta_i^t\, (\hat{v}_i - M_i) + \sum_{j \neq i} \gamma_{i,j}^t\, \tilde{p}_j^t \Big). \quad (4)
\end{aligned}$$

We consider each one of the three terms on the right side above one by one. For the first term, by Lemma 2, we have $G_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i}) = 0$. By the definition of M_i, we also have $\sum_{j \neq i} \gamma_{i,j}^t\, |\hat{p}_j^t - \tilde{p}_j^t| / \beta_i^t \leq M_i$ for all $t \in T$, so that $\sum_{t \in \mathcal{T}_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i})} \sum_{j \neq i} \gamma_{i,j}^t\, |\hat{p}_j^t - \tilde{p}_j^t| \leq M_i \sum_{t \in \mathcal{T}_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i})} \beta_i^t$. Thus, we get

$$\begin{aligned}
&\sum_{t \in \mathcal{T}_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i})} \Big(\alpha_i^t - \beta_i^t\, (\hat{v}_i - M_i) + \sum_{j \neq i} \gamma_{i,j}^t\, \tilde{p}_j^t \Big) - 2\, c_i \\
&\quad = \sum_{t \in \mathcal{T}_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i})} \Big(\alpha_i^t - \beta_i^t\, \hat{v}_i + \sum_{j \neq i} \gamma_{i,j}^t\, \hat{p}_j^t \Big) - 2\, c_i + \sum_{t \in \mathcal{T}_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i})} \sum_{j \neq i} \gamma_{i,j}^t\, (\tilde{p}_j^t - \hat{p}_j^t) + M_i \sum_{t \in \mathcal{T}_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i})} \beta_i^t \\
&\quad = G_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i}) + \sum_{t \in \mathcal{T}_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i})} \sum_{j \neq i} \gamma_{i,j}^t\, (\tilde{p}_j^t - \hat{p}_j^t) + M_i \sum_{t \in \mathcal{T}_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i})} \beta_i^t \\
&\quad \geq G_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i}) - \sum_{t \in \mathcal{T}_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i})} \sum_{j \neq i} \gamma_{i,j}^t\, |\tilde{p}_j^t - \hat{p}_j^t| + M_i \sum_{t \in \mathcal{T}_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i})} \beta_i^t \geq 0,
\end{aligned}$$

where the second equality uses the fact that $\hat{v}_i > M_i \geq 0$ so that we have $G_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i}) = \sum_{t \in \mathcal{T}_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i})} (\alpha_i^t - \beta_i^t\, \hat{v}_i + \sum_{j \neq i} \gamma_{i,j}^t\, \hat{p}_j^t) - 2\, c_i$. Therefore, the first term on the right side of (4) is non-negative. For the second term, by the definition of $\mathcal{T}_i(\hat{v}_i - M_i, \tilde{\boldsymbol{p}}_{-i})$, we have $\alpha_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, \tilde{p}_j^t > \beta_i^t\, (\hat{v}_i - M_i)$ for all $t \in \mathcal{T}_i(\hat{v}_i - M_i, \tilde{\boldsymbol{p}}_{-i})$. Therefore, we have $\mathbf{1}(t \in \mathcal{T}_i(\hat{v}_i - M_i, \tilde{\boldsymbol{p}}_{-i}) \setminus \mathcal{T}_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i}))\, (\alpha_i^t - \beta_i^t\, (\hat{v}_i - M_i) + \sum_{j \neq i} \gamma_{i,j}^t\, \tilde{p}_j^t) \geq 0$, which implies that the second term on the right side of (4) is non-negative. For the third term, we have $\alpha_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, \tilde{p}_j^t \leq \beta_i^t\, (\hat{v}_i - M_i)$ for all $t \notin \mathcal{T}_i(\hat{v}_i - M_i, \tilde{\boldsymbol{p}}_{-i})$. Therefore, we have $\mathbf{1}(t \in \mathcal{T}_i(\hat{v}_i, \hat{\boldsymbol{p}}_{-i}) \setminus \mathcal{T}_i(\hat{v}_i - M_i, \tilde{\boldsymbol{p}}_{-i})) \times (\alpha_i^t - \beta_i^t\, (\hat{v}_i - M_i) + \sum_{j \neq i} \gamma_{i,j}^t\, \tilde{p}_j^t) \leq 0$, indicating that the third term on the right side of (4) is non-positive. So, the first and second terms on the right side of (4) is non-negative, whereas the third term is non-positive, in which case, we have $G_i(\hat{v}_i - M_i, \tilde{p}_{-i}) \geq 0$. $\square$

For the vector $\boldsymbol{y} = \{y^t : t \in T\}$, we define the norm on $\Re^\tau$ as $\|\boldsymbol{y}\|_\infty = \max_{t \in T} |y^t|$. By the assumption that $\sum_{j \neq i} \gamma_{i,j}^t < \beta_i^t$ for all $i \in N$ and $t \in T$, Theorem 3 implies that the best response of firm i is a contraction mapping with respect to the norm $\| \cdot \|_\infty$, when viewed as a function of the prices of the other firms. Therefore, it immediately follows that if the price charged by each firm affects its demand more than the prices charged by the other firms, then there always exists a unique equilibrium without recourse.

3 Equilibrium with Recourse

In this section, we consider *strategies with recourse*, where each firm can change its price at the current time period as a function of its inventory and the inventories of the other firms. In other words, the firms do not commit to a price trajectory at the beginning of the selling horizon. We let x_i^t be the inventory of firm i at the beginning of time period t. Focusing on Markovian strategies without loss of generality, as a function of the inventories $\boldsymbol{x}^t = (x_1^t, \ldots, x_n^t)$ of all of the firms, we use $P_i^t(\boldsymbol{x}^t)$ to denote the price charged by firm i at time period t. It is useful to view $P_i^t(\cdot)$ as a function that determines the strategy of firm i at time period t as a function of the inventories of all of the firms. We use $\boldsymbol{P}^t = (P_1^t(\cdot), \ldots, P_n^t(\cdot))$ to capture the strategies of all of the firms at time period t and $\boldsymbol{P}_{-i}^t = (P_1^t(\cdot), \ldots, P_{i-1}^t(\cdot), P_{i+1}^t(\cdot), \ldots, P_n^t(\cdot))$ to capture the strategies of the firms other than firm i at time period t. If the firms other than firm i use the strategies $\boldsymbol{P}_{-i} = \{\boldsymbol{P}_{-i}^t : t \in T\}$, then we can find the best response strategy of firm i by solving the dynamic program

$$
\begin{aligned}
V_i^t(\boldsymbol{x}^t) = \max \Bigg\{ & \Big(\alpha_i^t - \beta_i^t\, p_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, P_j^t(\boldsymbol{x}^t)\Big)\, p_i^t + V_i^{t+1}(\boldsymbol{x}^{t+1}) \ :\ \alpha_i^t - \beta_i^t\, p_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, P_j^t(\boldsymbol{x}^t) \geq 0, \\
& x_i^{t+1} = x_i^t - \Big(\alpha_i^t - \beta_i^t\, p_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, P_j^t(\boldsymbol{x}^t)\Big), \\
& x_\ell^{t+1} = x_\ell^t - \Big(\alpha_\ell^t - \beta_\ell^t\, P_\ell^t(\boldsymbol{x}^t) + \gamma_{\ell,i}^t\, p_i^t + \sum_{j \notin \{i,\ell\}} \gamma_{\ell,j}^t\, P_j^t(\boldsymbol{x}^t)\Big) \ \forall\, \ell \in N \setminus \{i\}, \\
& p_i^t \geq 0,\ x_\ell^{t+1} \geq 0 \ \forall\, \ell \in N \Bigg\},
\end{aligned}
$$

with the boundary condition that $V_i^{\tau+1}(\cdot) = 0$. An optimal solution to the problem above characterizes the best response strategy of firm i at time period t.

If the strategy $\{P_i^t(\cdot) : t \in T\}$ chosen by each firm i is the best response to the strategy $\boldsymbol{P}_{-i}$ chosen by the other firms, then we say that the strategies $\{\boldsymbol{P}^t : t \in T\}$ chosen by the firms is an *equilibrium with recourse*. In the previous section, we show that there always exists a unique equilibrium when we focus on strategies without recourse. We give two numerical examples to show that if we focus on strategies with recourse, then there may not exist an equilibrium or there may be multiple equilibria. Consider the case where there are two firms and the selling horizon has two time periods. For given inventories of the two firms at the second time period, the problem of computing the equilibrium strategy at the second time period is identical to finding an equilibrium without recourse. So, there exists a unique equilibrium strategy for the firms at the second time period for given inventories. Note that the prices charged by the firms in an equilibrium without recourse at the second time period depends on the inventories of the firms at the second time period, which, in turn, depends on the prices charged by the firms at the first time period. To obtain an equilibrium with recourse, we compute the best response strategy of each firm at the first time period as a function of the price of the other firm at the first time period. Recall that if we fix the prices of the firms at the first time period, then we fix the inventories at the second time period,

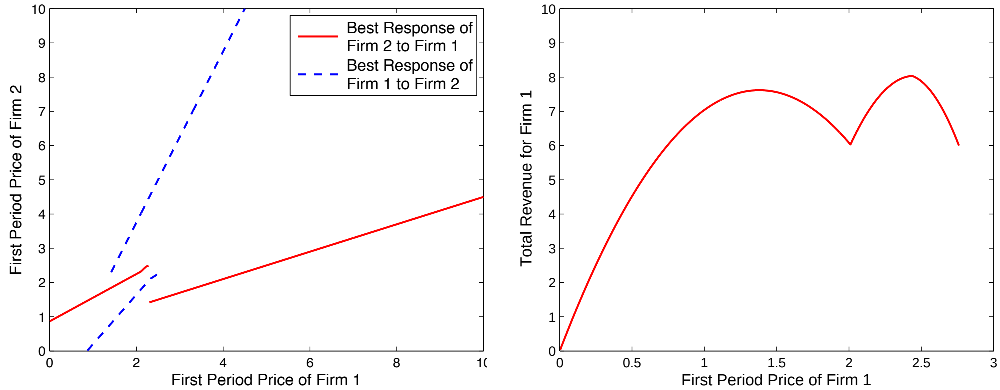

Figure 1: Best response of each firm when equilibrium with recourse does not exist and revenue of the first firm.

in which case, we can compute the equilibrium strategies at the second time period. We plot the best response of each firm at the first time period as a function of the price of the other firm. An equilibrium with recourse corresponds to the intersection of the two best response curves.

Consider the parameters $\alpha_i^t = 4$, $\beta_i^1 = 4$, $\beta_i^2 = 2$, $\gamma_{i,j}^1 = 16/5$, $\gamma_{i,j}^2 = 1$, $c_i = 3$ for all $i \in \{1, 2\}$, $j \in \{1, 2\} \setminus \{i\}$ and $t \in \{1, 2\}$, which satisfy $\sum_{j \neq i} \gamma_{i,t}^t < \beta_i^t$ for all $i \in \{1, 2, \}$ and $t \in \{1, 2\}$, so that we know that there exists a unique equilibrium without recourse. On the left side of Figure 1, the solid line plots the best response of second firm at the first time period on the vertical axis, as a function of the price of the first firm on the horizontal axis, whereas the dashed line plots the best response of the first firm at the first time period on the horizontal axis as a function of the price of the second firm on the vertical axis. The two best response functions do not intersect. Therefore, there does not exists an equilibrium with recourse. The main driver of the lack of equilibrium is the discontinuity in the best response function. The discontinuity is due to the fact that the revenue of each firm is a multi-modal function of its price at the first time period. On the right side of Figure 1, we show the revenue of the first firm as a function of its price at the first time period, when the price of the second firms is fixed at 2.2. So, firm 1 can jump from one mode to another based on the price of the second firm. Considering the parameters $\alpha_i^t = 4$, $\beta_i^1 = 5$, $\beta_i^2 = 2$, $\gamma_{i,j}^1 = 0.1$, $\gamma_{i,j}^2 = 1$ and $c_i = 5$ for all $i \in \{1, 2\}$, $j \in \{1, 2\} \setminus \{i\}$ and $t \in \{1, 2\}$, Figure 2 shows the best response of each firm at the first time period as a function of the price charged by the other firm. The best response functions intersect at two points, indicating multiple equilibria with recourse.

4 An Approximate Equilibrium

If the strategy $\{P_i^t(\cdot) : t \in T\}$ chosen by each firm i cannot increase the revenue of firm i by more than ϵ given the the other firms use the strategies $\boldsymbol{P}_{-i}$, then we say that the price strategies $\{\boldsymbol{P}^t : t \in T\}$ chosen by the firms is an *ϵ-equilibrium with recourse*. Since there may not exist an

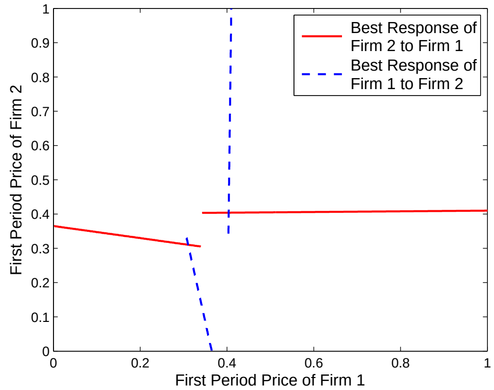

Figure 2: Best response of each firm when there are multiple equilibria with recourse.

equilibrium with recourse or there may be multiple equilibria with recourse, we focus on ϵ-equilibria with recourse. We consider a low influence regime, where, roughly speaking, the price charged by a firm affects its demand more than the prices charged by each of the other firms. In particular, letting $M = \max_{i \in N, t \in T} \sum_{j \neq i} \gamma^t_{i,j} / \beta^t_i$ and noting the assumption that $\sum_{j \neq i} \gamma^t_{i,j} < \beta^t_i$ for all $i \in N$, $t \in T$, we have $M < 1$. We consider the regime where the price charged by a firm affects its demand so much more than the prices charged by each of the other firms such that we have $\gamma^t_{i,j} / \beta^t_i < (1/M) - 1$. When $\sum_{j \neq i} \gamma^t_{i,j} / \beta^t_i < 1$ and the number of firms is large, we expect this assumption to hold. For example, if we have a symmetric setting, where the parameters related to each firm are the same, then under the assumption that $\sum_{j \neq i} \gamma^t_{i,j} < \beta^t_i$, we have $\gamma^t_{i,j} / \beta^t_i < 1/(n-1)$, in which case, the low influence regime naturally holds as the number of firms gets large. In the low influence regime, we show that the equilibrium without recourse studied in the previous section is an ϵ-equilibrium with recourse. Intuitively speaking, this result uses the fact that if $\gamma^t_{i,j} / \beta^t_i$ is small, then any deviation of a firm from a given price trajectory has little influence on the prices of the other firms in the subsequent time periods. In the next lemma, we formalize this idea. Throughout the rest of this section, we use $\mu = \max_{i \in N,\, j \in N \setminus \{i\},\, t \in T} \gamma^t_{i,j} / \beta^t_i$ and $\bar{\beta} = \max_{i \in N,\, t \in T} \beta^t_i / \min_{i \in N,\, t \in T} \beta^t_i$. Note that μ is expected to be small in the low influence regime.

Lemma 4 *Fixing the prices $\hat{\boldsymbol{p}}^1$ charged by the firms at the first time period, let the prices $\{\hat{\boldsymbol{p}}^t : t \in T \setminus \{1\}\}$ form the equilibrium without recourse in the remaining portion of the selling horizon. Define the prices $\tilde{\boldsymbol{p}}^1$ at the first time period as $\tilde{p}^1_i = \hat{p}^1_i + \delta$ and $\tilde{p}^1_j = \hat{p}_j$ for all $j \in N \setminus \{i\}$ for some $\delta \geq 0$. Fixing the prices $\tilde{\boldsymbol{p}}^1$ charged by the firms at the first time period, let the prices $\{\tilde{\boldsymbol{p}}^t : t \in T \setminus \{1\}\}$ form the equilibrium without recourse in the remaining portion of the selling horizon. If we have $\mu < (1/M) - 1$, then $\max_{j \neq i,\, t \in T \setminus \{1\}} |\hat{p}^t_j - \tilde{p}^t_j| \leq \frac{2\,\mu\,\bar{\beta}\,\delta}{1 - M - M\mu}$.*

Proof. We consider the problem over the time periods $T \setminus \{1\}$ where the inventory for each firm ℓ at the second time period is given by $c_\ell - D^1_\ell(\tilde{\boldsymbol{p}}^1)$. By the discussion that follows Theorem 3, if we start with any set of prices for the firms at the initial iteration and iteratively we compute the best response of each firm to prices at the previous iteration, then we reach the equilibrium without recourse. Therefore, to compute $\{\tilde{\boldsymbol{p}}^t : t \in T \setminus \{1\}\}$, we consider the problem over the time periods $T \setminus \{1\}$ with the inventory of each firm ℓ at the second time period given by $c_\ell - D^1_\ell(\tilde{\boldsymbol{p}}^1)$ and starting with the prices $\{\hat{\boldsymbol{p}}^t : t \in T \setminus \{1\}\}$ at the initial iteration, we iteratively compute the best response of each firm to the prices at the previous iteration. Letting $\{\tilde{\boldsymbol{p}}^{t,k} : t \in T \setminus \{1\}\}$ be the price trajectories for the firms at iteration k, we know that $\lim_{k\to\infty} \tilde{\boldsymbol{p}}^{t,k}_\ell = \tilde{\boldsymbol{p}}^t_\ell$ for all $\ell \in N$, $t \in T \setminus \{1\}$. Therefore, for all $t \in T \setminus \{1\}$ and $\ell \in N$, we have $|\hat{p}^t_\ell - \tilde{p}^t_\ell| = |\hat{p}^t_\ell - \tilde{p}^{t,1}_\ell + \sum_{k=1}^\infty (\tilde{p}^{t,k}_\ell - \tilde{p}^{t,k+1}_\ell)| \leq \sum_{k=1}^\infty |\tilde{p}^{t,k}_\ell - \tilde{p}^{t,k+1}_\ell|$, where the inequality uses the fact that $\tilde{p}^{t,1}_\ell = \hat{p}^t_\ell$. In this case, to bound $|\hat{p}^t_\ell - \tilde{p}^t_\ell|$, we can bound $|\tilde{p}^{t,k}_\ell - \tilde{p}^{t,k+1}_\ell|$ for all $k = 1, 2, \ldots$ and add up the bounds on the latter quantity. We proceed to bounding $|\tilde{p}^{t,k}_\ell - \tilde{p}^{t,k+1}_\ell|$. By definition, the price trajectory $\{\tilde{p}^{t,k+1}_\ell : t \in T \setminus \{1\}\}$ of firm ℓ at iteration $k+1$ is the best response of firm ℓ to the price trajectories $\{\tilde{\boldsymbol{p}}^{t,k}_{-\ell} : t \in T \setminus \{1\}\}$ of the other firms at iteration k. In this case, Theorem 3 implies that $|\tilde{p}^{t,k+1}_\ell - \tilde{p}^{t,k}_\ell| \leq \max_{t \in T\setminus\{1\}} \{\sum_{j \neq \ell} \gamma^t_{\ell,j}\, |\tilde{p}^{t,k}_j - \tilde{p}^{t,k-1}_j| / \beta^t_\ell\}$ for all $\ell \in N$, $t \in T \setminus \{1\}$, $k = 2, 3, \ldots$. For notational brevity, we let $\Phi^k_\ell = \max_{t \in T\setminus\{1\}} \{|\tilde{p}^{t,k+1}_\ell - \tilde{p}^{t,k}_\ell|\}$ so that the last inequality yields $\Phi^k_\ell \leq \max_{t \in T\setminus\{1\}} \sum_{j\neq\ell} \gamma^t_{\ell,j}\, \Phi^{k-1}_j / \beta^t_\ell$ for all $\ell \in N$, $k = 2, 3, \ldots$. Using the inequality $\Phi^k_\ell \leq \max_{t \in T\setminus\{1\}} \sum_{j\neq\ell} \gamma^t_{\ell,j}\, \Phi^{k-1}_j / \beta^t_\ell$ for firm $\ell = j$ with $j \neq i$ and noting the definitions of M and μ, it follows that

$$\Phi^k_j \leq \max_{t \in T\setminus\{1\}} \left\{ \frac{\sum_{\ell \in N\setminus\{j,i\}} \gamma^t_{j,\ell}\, \Phi^{k-1}_\ell}{\beta^t_j} + \frac{\gamma^t_{j,i}\, \Phi^{k-1}_i}{\beta^t_j} \right\} \leq M \max_{\ell \in N\setminus\{j,i\}} \left\{ \Phi^{k-1}_\ell \right\} + \mu\, \Phi^{k-1}_i \tag{5}$$

for all $j \neq i$ and $k = 2, 3, \ldots$. Using the inequality $\Phi^k_\ell \leq \max_{t \in T\setminus\{1\}} \sum_{j\neq\ell} \gamma^t_{\ell,j}\, \Phi^{k-1}_j / \beta^t_\ell$ again for firm $\ell = i$, we get $\Phi^k_i \leq \max_{t \in T\setminus\{1\}} \sum_{j \neq i} \gamma^t_{i,j}\, \Phi^{k-1}_j / \beta^t_i \leq M\, \max_{j\neq i} \Phi^{k-1}_j$ for all $k = 2, 3, \ldots$. If we use the last inequality in (5), then for all $j \neq i$ and $k = 3, 4, \ldots$, we have $\Phi^k_j \leq M\, \max_{\ell \in N\setminus\{j,i\}} \{\Phi^{k-1}_\ell\} + M\mu\, \max_{j\neq i} \{\Phi^{k-2}_j\}$. So, letting $\Theta^k = \max_{j \neq i} \Phi^k_j$, the last inequality yields

$$\Theta^k \leq M \Theta^{k-1} + M\mu\, \Theta^{k-2}$$

for all $k = 3, 4, \ldots$. Adding the inequality above over all $k = 3, 4, \ldots$, we obtain $\sum_{k=3}^\infty \Theta^k \leq M \sum_{k=2}^\infty \Theta^k + M\mu \sum_{k=1}^\infty \Theta^k$, which is equivalent to $\sum_{k=1}^\infty \Theta^k \leq M \sum_{k=2}^\infty \Theta^k + M\mu \sum_{k=1}^\infty \Theta^k + \Theta^1 + \Theta^2 = (M + M\mu) \sum_{k=1}^\infty \Theta^k + (1 - M)\, \Theta^1 + \Theta^2$. Rearranging the terms in the last chain of inequalities, we get $\sum_{k=1}^\infty \Theta^k \leq ((1 - M)\, \Theta^1 + \Theta^2)/(1 - M - M\mu)$.

Therefore, if we can bound Θ^1 and Θ^2, then we can bound $\sum_{k=1}^\infty \Theta^k$. When we increase the price of firm i at the first time period by δ, the inventory of firm i at the second time period changes by at most $\beta^1_i\, \delta$ and the inventory of firm $j \neq i$ at the second time period changes by at most $\gamma^1_{j,i}\, \delta$. In the appendix, Lemma 8 shows that if we fix the price trajectories of the firms other than firm i, then the best response of firm i, when viewed as a function of its initial inventory, is Lipschitz with constant $1/\beta_{\min}$, where we let $\beta_{\min} = \min_{i \in N,\ t \in T} \beta^t_i$. Note that the best response of firm i does not depend on the inventories of the other firms, since the price trajectories of the other firms is fixed. By definition, if we consider the problem over the time periods $T \setminus \{1\}$ with the inventory of each firm ℓ at the second time period given by $c_\ell - D^1_\ell(\hat{\boldsymbol{p}}^1)$, by definition, $\{\hat{\boldsymbol{p}}^t_\ell : t \in T \setminus \{1\}\}$ is the best response to the price trajectories $\{\hat{\boldsymbol{p}}^t_{-\ell} : t \in T \setminus \{1\}\}$. Also, if we consider the problem over the time periods $T\setminus\{1\}$ with the inventory of each firm ℓ at the second time period given by $c_\ell - D^1_\ell(\tilde{\boldsymbol{p}}^1)$, by definition, $\{\tilde{\boldsymbol{p}}^{t,2}_\ell : t \in T \setminus \{1\}\}$ is the best response to the price trajectories $\{\tilde{\boldsymbol{p}}^{t,1}_{-\ell} : t \in T \setminus \{1\}\}$. Since the price trajectories $\{\hat{\boldsymbol{p}}^t_{-\ell} : t \in T \setminus \{1\}\}$ and $\{\tilde{\boldsymbol{p}}^{t,1}_{-\ell} : t \in T \setminus \{1\}\}$ are the same, Lemma 8 in the appendix implies that $|\tilde{p}^{t,2}_\ell - \tilde{p}^{t,1}_\ell| = |\tilde{p}^{t,2}_\ell - \hat{p}^t_\ell| \leq |(c_\ell - D^1_\ell(\tilde{\boldsymbol{p}}^1)) - (c_\ell - D^1_\ell(\hat{\boldsymbol{p}}^1))|/\beta_{\min}$ for all $t \in T \setminus \{1\}$. As discussed at the beginning of this paragraph, the expression on the right side of the last inequality is bounded by $\beta^1_i\,\delta$ when $\ell = i$ and bounded by $\gamma^1_{j,i}\,\delta$ when $\ell = j$ with $j \neq i$. Therefore, we obtain $|\tilde{p}^{t,2}_i - \tilde{p}^{t,1}_i| \leq \beta^1_i \delta/\beta_{\min} \leq \bar{\beta}\,\delta$ and $|\tilde{p}^{t,2}_j - \tilde{p}^{t,1}_j| \leq \gamma^1_{j,i}\,\delta/\beta_{\min} \leq \bar{\beta}\,\mu\,\delta$ for all $j \neq i$. The second one of the last two inequalities yields $\Theta^1 = \max_{j\neq i,\; t\in T\setminus\{1\}}\{|\tilde{p}^{t,2}_j - \tilde{p}^{t,1}_j|\} \leq \mu\,\bar{\beta}\,\delta$. The first one of the last two inequalities yields $\Phi^1_i = \max_{t\in T\setminus\{1\}}\{|\tilde{p}^{t,2}_i - \tilde{p}^{t,1}_i|\} \leq \bar{\beta}\,\delta$, in which case, noting (5), we get $\Theta^2 = \max_{j\neq i}\{\Phi^2_j\} \leq M\,\Theta^1 + \mu\,\Phi^1_i \leq M\mu\,\bar{\beta}\,\delta + \mu\,\bar{\beta}\,\delta$. Thus, Θ^1 and Θ^2 are respectively bounded by $\mu\,\bar{\beta}\,\delta$ and $(1+M)\,\mu\,\bar{\beta}\,\delta$. In this case, for all $j \neq i$ and $t \in T \setminus \{1\}$, we have

$$
\begin{aligned}
|\hat{p}^t_j - \tilde{p}^t_j| &\leq \sum_{k=1}^{\infty} |\tilde{p}^{t,k+1}_j - \tilde{p}^{t,k}_j| \leq \sum_{k=1}^{\infty} \max_{j\neq i, t\in T\setminus\{1\}} \left\{|\tilde{p}^{t,k+1}_j - \tilde{p}^{t,k}_j|\right\} = \sum_{k=1}^{\infty} \Theta^k \\
&\leq \frac{(1-M)\,\Theta^1 + \Theta^2}{1-M-M\mu} \leq \frac{(1-M)\,\mu\,\bar{\beta}\,\delta + (1+M)\,\mu\,\bar{\beta}\,\delta}{1-M-M\mu} \leq \frac{2\,\mu\,\bar{\beta}\,\delta}{1-M-M\mu},
\end{aligned}
$$

where the first inequality follows from the discussion at the beginning of the proof and the equality is by the definition of Θ^k and Φ^k_ℓ. □

Consider the problem over the time periods $\kappa, \ldots, \tau$ when the inventories of the firms at time period κ are given by $\boldsymbol{x} = (x_1, \ldots, x_n)$. We use $p^{N,t}_i(\kappa, \boldsymbol{x})$ to denote the price charged by firm i at time period t in the equilibrium without recourse. We consider the following strategy with recourse for firm i. If the inventories of the firms at time period t is given by $\boldsymbol{x}$, then firm i charges the price $p^{N,t}_i(t, \boldsymbol{x})$. In other words, letting $P^{R,t}_i(\cdot)$ be the strategy function of firm under this strategy with recourse, we have $P^{R,t}_i(\boldsymbol{x}) = p^{N,t}_i(t, \boldsymbol{x})$. Using $\boldsymbol{P}^{R,t} = (P^{R,t}_1(\cdot), \ldots, P^{R,t}_n(\cdot))$ to capture the strategies of all of the firms at time period t and $\boldsymbol{c} = (c_1, \ldots, c_n)$ to denote the inventories of the firms at the first time period, note that if all firms use the strategies $\{\boldsymbol{P}^{R,t} : t \in T\}$ over the selling horizon, then the price charged by each firm i at each time period t is given by $p^{N,t}_i(1, \boldsymbol{c})$, which is precisely the prices corresponding to the equilibrium without recourse when we consider the problem over the time periods T with the inventories of the firms at the first time period given by $\boldsymbol{c}$. However, if one of the firms deviate from the strategies $\{\boldsymbol{P}^{R,t} : t \in T\}$ at a time period, then the prices charged by the firms will be different from those in the equilibrium without recourse. Therefore, it is not generally true that the strategies $\{\boldsymbol{P}^{R,t} : t \in T\}$ correspond to an equilibrium with recourse. In the remainder of this section, we show that the strategies $\{\boldsymbol{P}^{R,t} : t \in T\}$ correspond to an ϵ-equilibrium without recourse in the low influence regime. In the next lemma, we show that if firm i unilaterally deviates from the strategy $\{P^{R,t}_i(\cdot) : t \in T\}$, but the other firms use the strategies $\{\boldsymbol{P}^{R,t} : t \in T\}$, then firm i does not increase its revenue by more than a simple function of μ.

In the proof of the next lemma, we make use of the fact that there is a natural upper bound on the prices that can be used by each firm. As discussed in Section 2, we restrict the strategy space of the firms such that $\alpha_i^t - \beta_i^t\, p_i^t + \sum_{j\neq i} \gamma_{i,j}^t\, p_j^t \geq 0$ for all $i \in N$. Noting the definition of M, for all $i \in N$, we get $p_i^t \leq \alpha_i^t/\beta_i^t + \sum_{j\neq i} \gamma_{i,j}^t\, p_j^t/\beta_i^t \leq \max_{i\in N}\{\alpha_i^t/\beta_i^t\} + \max_{i\in N}\{\sum_{j\neq i} \gamma_{i,j}^t/\beta_i^t\} \times \max_{j\in N}\{p_j^t\} \leq \max_{i\in N}\{\alpha_i^t/\beta_i^t\} + M \max_{j\in N}\{p_j^t\}$, which implies that $\max_{i\in N}\{p_i^t\} \leq \max_{i\in N}\{\alpha_i^t/\beta_i^t\} + M \max_{j\in N}\{p_j^t\}$. In this case, we obtain the upper bound on the maximum price given by $\max_{i\in N}\{p_i^t\} \leq \max_{i\in N}\{\alpha_i^t/\beta_i^t\}/(1-M)$.

Lemma 5 *Assume that the strategies of all of the firms are $\{\boldsymbol{P}^{R,t} : t \in T\}$. Let Π_i^N be the revenue of firm i under these strategies. Also, assume that the strategies of the firms other than firm i are $\{\boldsymbol{P}_{-i}^{R,t} : t \in T\}$, but firm i deviates to charge an arbitrary price at the first time period and uses the strategy $\{P_i^{R,t}(\cdot) : t \in T \setminus \{1\}\}$ at the other time periods. Let Π_i^D be the revenue of firm i under this strategy. Letting $P_{\max} = \max_{i\in N}\{\alpha_i^t/\beta_i^t\}/(1-M)$ and $\beta_{\max} = \max_{i\in N, t\in T} \beta_i^t$, we have*

$$\Pi_i^D - \Pi_i^N \leq \frac{2\,\bar\beta\, M \beta_{\max}\, P_{\max}^2\, (\tau - 1)\, \mu}{1 - M - M\mu}.$$

Proof. We let $\hat p_i^t$ be the price charged by firm i at time period t in the equilibrium without recourse. As discussed right before the lemma, given that all of the firms use the strategy $\{\boldsymbol{P}^{R,t} : t \in T\}$, the realized prices are $\{\hat p_i^t : i \in N,\ t \in T\}$. We use $\hat q_i^1$ to denote the arbitrary price charged by firm i at the first time period. Given that firm i uses the strategy $\{P_i^{R,t}(\cdot) : t \in T\setminus\{1\}\}$ at the other time periods and the other firms use the strategy $\{\boldsymbol{P}_{-i}^{R,t} : t \in T\}$, we let $\{\hat q_i^t : i \in N,\ t \in T\}$ be the realized prices. For each firm $j \neq i$, note that $\hat q_j^1 = P_j^{R,1}(\boldsymbol{c}) = p_j^{N,1}(1, \boldsymbol{c}) = \hat p_j^1$. Also, by Lemma 4, for all $j \neq i$ and $t \in T \setminus \{1\}$, we have $|\hat p_j^t - \hat q_j^t| \leq 2\,\mu\,\bar\beta\,|\hat p_i^1 - \hat q_i^1|/(1 - M - M\mu) \leq 2\,\mu\,\bar\beta\, P_{\max}/(1 - M - M\mu)$. We use $\pi_i^t(p_i^t, \boldsymbol{p}_{-i}^t)$ to denote the revenue of firm i at time period t when firm i charges the price p_i^t and the other firms charge the price $\boldsymbol{p}_{-i}^t = (p_1^t, \ldots, p_{i-1}^t, p_{i+1}^t, \ldots, p_n^t)$. We have $\Pi_i^N = \sum_{t\in T} \pi_i^t(\hat p_i^t, \hat{\boldsymbol{p}}_{-i}^t)$ and $\Pi_i^D = \sum_{t\in T} \pi_i^t(\hat q_i^t, \hat{\boldsymbol{q}}_{-i}^t)$. In this case, we get

$$\Pi_i^N = \sum_{t\in T} \pi_i^t(\hat p_i^t, \hat{\boldsymbol{p}}_{-i}^t) \geq \sum_{t\in T} \pi_i^t(\hat q_i^t, \hat{\boldsymbol{p}}_{-i}^t) = \Pi_i^D - \sum_{t\in T\setminus\{1\}} \pi_i^t(\hat q_i^t, \hat{\boldsymbol{q}}_{-i}^t) + \sum_{t\in T\setminus\{1\}} \pi_i^t(\hat q_i^t, \hat{\boldsymbol{p}}_{-i}^t),$$

where the inequality uses the fact that $\{\hat p_i^t : t \in T\}$ is the best response of firm i to the prices $\{\hat{\boldsymbol{p}}_{-i}^t : t \in T\}$ and the second equality uses the fact that $\hat{\boldsymbol{q}}_{-i}^1 = \hat{\boldsymbol{p}}_{-i}^1$. Using $D_i^t(p_i^t, \boldsymbol{p}_{-i}^t)$ to denote the demand of firm i at time period t when firm i charges the price p_i^t and the other firms charge the prices $\boldsymbol{p}_{-i}^t$, by the inequality above, we get $\Pi_i^D - \Pi_i^N \leq \sum_{t\in T\setminus\{1\}} |\pi_i^t(\hat q_i^t, \hat{\boldsymbol{q}}_{-i}^t) - \pi_i^t(\hat q_i^t, \hat{\boldsymbol{p}}_{-i}^t)| = \sum_{t\in T\setminus\{1\}} |\hat q_i^t\, D_i^t(\hat q_i^t, \hat{\boldsymbol{q}}_{-i}^t) - \hat q_i^t D_i^t(\hat q_i^t, \hat{\boldsymbol{p}}_{-i}^t)|$. Since $D_i^t(\hat q_i^t, \hat{\boldsymbol{q}}_{-i}^t) - D_i^t(\hat q_i^t, \hat{\boldsymbol{p}}_{-i}^t) = (\alpha_i^t + \beta_i^t\, \hat q_i^t - \sum_{j\neq i} \gamma_{i,j}^t\, \hat q_j^t) - (\alpha_i^t + \beta_i^t\, \hat q_i^t - \sum_{j\neq i} \gamma_{i,j}^t\, \hat p_j^t) = \sum_{j\neq i} \gamma_{i,j}^t\, (\hat p_j^t - \hat q_j^t)$, the last chain of inequalities yields $\Pi_i^D - \Pi_i^N \leq \sum_{t\in T\setminus\{1\}} \hat q_i^t \sum_{j\neq i} \gamma_{i,j}^t\, |\hat q_j^t - \hat p_j^t| \leq \sum_{t\in T\setminus\{1\}} P_{\max}\, M \beta_i^t \max_{j\neq i} |\hat q_j^t - \hat p_j^t|$, where we use the fact that $\hat q_i^t \leq P_{\max}$ and $M \geq \sum_{j\neq i} \gamma_{i,j}^t/\beta_i^t$. At the beginning of the proof, we show that $|\hat p_j^t - \hat q_j^t| \leq 2\,\mu\,\bar\beta\, P_{\max}/(1 - M - M\mu)$ for all $j \neq i$ and $t \in T \setminus \{1\}$. In this case, we obtain $\Pi_i^D - \Pi_i^N \leq \sum_{t\in T\setminus\{1\}} P_{\max}\, M \beta_i^t \max_{j\neq i}\{|\hat q_j^t - \hat p_j^t|\} = 2\,\bar\beta M \beta_{\max} P_{\max}^2\, (\tau - 1)\, \mu/(1 - M - M\mu)$. ☐

In the next theorem, we show that the strategy $\{\boldsymbol{P}^{R,t} : t \in T\}$ is an ϵ-equilibrium with recourse, when the number of firms is large so that μ is expected to be small.

Theorem 6 *Assume that the strategies of all of the firms are $\{\boldsymbol{P}^{R,t} : t \in T\}$. Let Π_i^N be the revenue of firm i under these strategies. Also, assume that the strategy of the firms other than firm i are $\{\boldsymbol{P}_{-i}^{R,T} : t \in T\}$, but firm i uses an arbitrary strategy over the whole selling horizon. Let Π_i^A be the revenue of firm i under these strategies. Letting $\Gamma_\mu = \bar{\beta}\, M \beta_{\max}\, P_{\max}^2/(1 - M - M\mu)$, we have*

$$\Pi_i^A - \Pi_i^N \le \Gamma_\mu\, \tau\, (\tau - 1)\, \mu.$$

Proof. Consider the problem over the time periods $\kappa, \ldots, \tau$. We use $\mathsf{Rev}_i^\kappa(P_i^\kappa(\cdot), \ldots, P_i^\tau(\cdot), \boldsymbol{P}_{-i}, \boldsymbol{x})$ to denote the revenue of firm i over the time periods $\kappa, \ldots, \tau$, when the firm uses the strategy $\{P_i^\kappa(\cdot), \ldots, P_i^\tau(\cdot)\}$, the other firms use the strategy $\boldsymbol{P}_{-i}$ and the inventories at time period κ are given by $\boldsymbol{x}$. We let $\{Q_i^t(\cdot) : t \in T\}$ be an arbitrary strategy used by firm i. We use induction over the time periods to show that $\mathsf{Rev}_i^\kappa(Q_i^\kappa(\cdot), \ldots, Q_i^\tau(\cdot), \boldsymbol{P}_{-i}^R, \boldsymbol{x}) - \mathsf{Rev}_i^\kappa(P_i^{R,\kappa}(\cdot), \ldots, P_i^{R,\tau}(\cdot), \boldsymbol{P}_{-i}^R, \boldsymbol{x}) \le \Gamma_\mu\, (\tau - \kappa + 1)\, (\tau - \kappa)\, \mu$. In this case, the result follows by noting that $\Pi_i^A = \mathsf{Rev}_i^1(Q_i^1(\cdot), \ldots, Q_i^\tau(\cdot), \boldsymbol{P}_{-i}^R, \boldsymbol{x})$, $\Pi_i^N = \mathsf{Rev}_i^1(P_i^{R,1}(\cdot), \ldots, P_i^{R,\tau}(\cdot), \boldsymbol{P}_{-i}^R, \boldsymbol{x})$ and using the last inequality with $\kappa = 1$. Consider the case $\kappa = \tau$. We have $\mathsf{Rev}_i^\tau(Q_i^\tau(\cdot), \boldsymbol{P}_{-i}^R, \boldsymbol{x}) - \mathsf{Rev}_i^\tau(P_i^{R,\tau}(\cdot), \boldsymbol{P}_{-i}^R, \boldsymbol{x}) \le 0$, where the inequality follows form the fact that $P_i^{R,\tau}(\boldsymbol{x})$ is the best response of firm i to the prices $\boldsymbol{P}_{-i}^{R,t}(\boldsymbol{x})$. Therefore the result holds for $\kappa = \tau$. Assuming that the result holds for $\kappa = t + 1$, we show that the result holds for $\kappa = t$. Using $\boldsymbol{D}^t(p_i^t, \boldsymbol{p}_{-i}^t)$ to denote the vector of demands for the firms when the prices are $(p_i^t, \boldsymbol{p}_{-i}^t)$ and letting $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{D}^t(Q_i^t(\boldsymbol{x}), \boldsymbol{P}_{-i}^t(\boldsymbol{x}))$, observe that $\mathsf{Rev}_i^t(Q_i^t(\cdot), Q_i^{t+1}(\cdot), \ldots, Q_i^\tau(\cdot), \boldsymbol{P}_{-i}^R, \boldsymbol{x}) - \mathsf{Rev}_i^t(Q_i^t(\cdot), P_i^{R,t+1}(\cdot), \ldots, P_i^{R,\tau}(\cdot), \boldsymbol{P}_{-i}^R, \boldsymbol{x}) = \mathsf{Rev}_i^{t+1}(Q_i^{t+1}(\cdot), \ldots, Q_i^\tau(\cdot), \boldsymbol{P}_{-i}^R, \boldsymbol{x}') - \mathsf{Rev}_i^{t+1}(P_i^{R,t+1}(\cdot), \ldots, R_i^{R,\tau}(\cdot), \boldsymbol{P}_{-i}^R, \boldsymbol{x}')$, since all firms make the same pricing decisions at time period t in the two revenue expressions on the left side of the equality. By the induction assumption, the right side of the last equality is bounded by $\Gamma_\mu\, (\tau - t)\, (\tau - t - 1)\, \mu$. Also, considering $\mathsf{Rev}_i^t(Q_i^t(\cdot), P_i^{R,t+1}(\cdot), \ldots, R_i^{R,\tau}(\cdot), \boldsymbol{P}_{-i}^R, \boldsymbol{x}) - \mathsf{Rev}_i^t(P_i^{R,t}(\cdot), P_i^{R,t+1}(\cdot) \ldots, P_i^{R,\tau}(\cdot), \boldsymbol{P}_{-i}^R, \boldsymbol{x})$, this expression is the change in the revenue of firm i when firm i deviates from the strategy $\{P_i^{R,t}(\cdot) : t \in T\}$ only at the initial period and there are $\tau - t + 1$ time periods in the problem. By Lemma 5, this expression is bounded by $2\,\Gamma_\mu\, (\tau - t)\, \mu$. In this case, noting that

$$\begin{aligned}
&\mathsf{Rev}_i^t(Q_i^t(\cdot), \ldots, Q_i^\tau(\cdot), \boldsymbol{P}_{-i}^R, \boldsymbol{x}) - \mathsf{Rev}_i^t(P_i^{R,t}(\cdot), \ldots, P_i^{R,\tau}(\cdot), \boldsymbol{P}_{-i}^R, \boldsymbol{x}) \\
&\quad = \mathsf{Rev}_i^t(Q_i^t(\cdot), Q_i^{t+1}(\cdot), \ldots, Q_i^\tau(\cdot), \boldsymbol{P}_{-i}^R, \boldsymbol{x}) - \mathsf{Rev}_i^t(Q_i^t(\cdot), P_i^{R,t+1}(\cdot), \ldots, R_i^{R,\tau}(\cdot), \boldsymbol{P}_{-i}^R, \boldsymbol{x}) \\
&\qquad\quad + \mathsf{Rev}_i^t(Q_i^t(\cdot), P_i^{R,t+1}(\cdot), \ldots, R_i^{R,\tau}(\cdot), \boldsymbol{P}_{-i}^R, \boldsymbol{x}) - \mathsf{Rev}_i^t(P_i^{R,t}(\cdot), \ldots, P_i^{R,\tau}(\cdot), \boldsymbol{P}_{-i}^R, \boldsymbol{x}),
\end{aligned}$$

the two differences on the right side above are bounded by $\Gamma_\mu\, (\tau - t)\, (\tau - t + 1)\mu$ and $2\,\Gamma_\mu\, (\tau - t)\, \mu$. Since $\Gamma_\mu\, (\tau - t)\, (\tau - t - 1)\mu + 2\,\Gamma_\mu\, (\tau - t)\, \mu = \Gamma_\mu\, (\tau - t + 1)\, (\tau - t)\, \mu$, the result holds for $\kappa = t$. □

We observe that as μ approaches zero, $\Gamma_\mu\, \tau\, (\tau - 1)\, \mu$ approaches zero as well. Therefore, by the theorem above, if we are in the low influence regime, then no firm can improve its revenue significantly by deviating from the policy $\{\boldsymbol{P}^{R,t} : t \in T\}$, which implies that $\{\boldsymbol{P}^{R,t} : t \in T\}$ is an ϵ-equilibrium with recourse. As discussed earlier, the price trajectory realized under the strategy $\{\boldsymbol{P}^{R,t} : t \in T\}$ is precisely the price trajectory of the equilibrium without recourse.

5 Conclusions

We studied a competitive pricing problem. In equilibrium without recourse, each firm commits to a price trajectory, whereas in equilibrium with recourse, each firm can adjust its price at the current time period based on the inventories of all of the firms. Although the demand is a deterministic function of the prices so that there is no uncertainty in the responses of the firms, we showed that the two equilibrium concepts can be quite different. While the equilibrium without recourse always uniquely exists, the equilibrium with recourse may not exist or may not be unique. Our uniqueness proof for the equilibrium without recourse uses a contraction property. A natural research direction is to extend such contraction properties to demand models other than the linear demand model. Also, it would be useful to see whether an analogue of equilibrium without recourse can be defined under stochastic demand and check whether it uniquely exists.

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A Appendix: Omitted Results

In the following lemma, we show elementary properties for the function $G_i(\cdot, \boldsymbol{p}_{-i})$. We use these properties throughout the paper.

Lemma 7 *For fixed $\boldsymbol{p}_{-i}$, let $\nu^* = \inf\{\nu \in \Re_+ : \mathcal{T}_i(\nu, \boldsymbol{p}_{-i}) = \varnothing\}$. The function $G_i(\cdot, \boldsymbol{p}_{-i})$ satisfies the following properties.*

(a) The function $G_i(\nu, \boldsymbol{p}_{-i})$ is continuous in $\nu \in (0, \infty)$.

(b) The function $G_i(\nu, \boldsymbol{p}_{-i})$ is strictly decreasing in $\nu \in [0, \nu^*)$ and constant in $\nu \in [\nu^*, \infty)$ satisfying $G_i(\nu, \boldsymbol{p}_{-i}) = -2\, c_i$ for all $\nu \in [\nu^*, \infty)$.

(c) There exists unique $\hat{\nu} \in [0, \infty)$ satisfying $G_i(\hat{\nu}, \boldsymbol{p}_{-i}) = 0$.

Proof. First, we show Part a. Fix $\nu > 0$ and $\epsilon > 0$ small enough that $\nu - \epsilon > 0$. The definition of $\mathcal{T}_i(\nu, \boldsymbol{p}_{-i})$ implies that $\mathcal{T}_i(\nu - \epsilon, \boldsymbol{p}_{-i}) \supseteq \mathcal{T}_i(\nu, \boldsymbol{p}_{-i})$. Also, if $t \in \mathcal{T}_i(\nu - \epsilon, \boldsymbol{p}_{-i}) \setminus \mathcal{T}_i(\nu, \boldsymbol{p}_{-i})$, then we have $\nu \geq \frac{\alpha_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t}{\beta_i^t} > \nu - \epsilon$, which implies that we have $0 \geq \alpha_i^t - \beta_i^t\, \nu + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t > -\beta_i^t\, \epsilon$. For notational brevity, we let $\mathcal{T}_i^+ = \mathcal{T}_i(\nu, \boldsymbol{p}_{-i})$, $\mathcal{T}_i^- = \mathcal{T}_i(\nu - \epsilon, \boldsymbol{p}_{-i})$ and $\mathcal{U}_i = \mathcal{T}_i^- \setminus \mathcal{T}_i^+$ so that $\mathcal{T}_i^- = \mathcal{T}_i^+ \cup \mathcal{U}_i$. Noting the definition of $G_i(\nu, \boldsymbol{p}_{-i})$, we have

$$
\begin{aligned}
&G_i(\nu, \boldsymbol{p}_{-i}) - G_i(\nu - \epsilon, \boldsymbol{p}_{-i}) \\
&\quad = \left\{ \sum_{t \in \mathcal{T}_i^+} \Big(\alpha_i^t - \beta_i^t\, \nu + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t \Big) - 2\, c_i \right\} - \left\{ \sum_{t \in \mathcal{T}_i^-} \Big(\alpha_i^t - \beta_i^t\, (\nu - \epsilon) + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t \Big) - 2\, c_i \right\} \\
&\quad = - \sum_{t \in \mathcal{U}_i} \Big(\alpha_i^t - \beta_i^t\, \nu + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t \Big) - \sum_{t \in \mathcal{T}_i^+ \cup\, \mathcal{U}_i} \beta_i^t\, \epsilon,
\end{aligned}
$$

Since $0 \geq \alpha_i^t - \beta_i^t\, \nu + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t > -\beta_i^t\, \epsilon$ for all $t \in \mathcal{U}_i$, the equality above yields $-\sum_{t \in \mathcal{T}_i^+ \cup\, \mathcal{U}_i} \beta_i^t\, \epsilon \leq G_i(\nu, \boldsymbol{p}_{-i}) - G_i(\nu - \epsilon, \boldsymbol{p}_{-i}) \leq -\sum_{t \in \mathcal{T}_i^+} \beta_i^t\, \epsilon$, so that $G_i(\nu, \boldsymbol{p}_{-i})$ is continuous in $\nu \in (0, \infty)$.

Second, we show Part b. Fix $\nu \in (0, \nu^*)$, in which case, by the definition of ν^*, we have $\mathcal{T}_i(\nu, \boldsymbol{p}_{-i}) \neq \varnothing$. In the proof of Part a, we show that $G_i(\nu, \boldsymbol{p}_{-i}) - G_i(\nu - \epsilon, \boldsymbol{p}_{-i}) \leq -\sum_{t \in \mathcal{T}_i(\nu, \boldsymbol{p}_{-i})} \beta_i^t\, \epsilon$ for all $\epsilon > 0$ small enough that $\nu - \epsilon > 0$. Since $\beta_i^t > 0$ for all $t \in T$ and $\mathcal{T}_i(\nu, \boldsymbol{p}_{-i}) \neq \varnothing$, the last inequality implies that $G_i(\nu - \epsilon, \boldsymbol{p}_{-i}) > G_i(\nu, \boldsymbol{p}_{-i})$ for all $\nu \in (0, \nu^*)$ and $\epsilon > 0$ small enough that $\nu - \epsilon > 0$. Also, noting that $\alpha_i^t > 0$ and $\beta_i^t > 0$, by the definition of $\mathcal{T}_i(\nu, \boldsymbol{p}_{-i})$, we have $\mathcal{T}_i(\epsilon, \boldsymbol{p}_{-i}) = T$ for small enough $\epsilon > 0$. In this case, by the definition of $G_i(\nu, \boldsymbol{p}_{-i})$, we obtain $G_i(0, \boldsymbol{p}_{-i}) \geq \sum_{t \in T} (\alpha_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t) - 2\, c_i > \sum_{t \in T} (\alpha_i^t - \beta_i^t\, \epsilon + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t) - 2\, c_i = G_i(\epsilon, \boldsymbol{p}_{-i})$, which implies that $G_i(0, \boldsymbol{p}_{-i}) > G_i(\epsilon, \boldsymbol{p}_{-i})$ for small enough $\epsilon > 0$. Therefore, we have $G_i(\nu - \epsilon, \boldsymbol{p}_{-i}) > G_i(\nu, \boldsymbol{p}_{-i})$ for all $\nu \in (0, \nu^*)$ and $\epsilon > 0$ small enough that $\nu - \epsilon > 0$. Also, we have $G_i(0, \boldsymbol{p}_{-i}) > G_i(\epsilon, \boldsymbol{p}_{-i})$ for small enough $\epsilon > 0$. The last two statements establish that $G_i(\nu, \boldsymbol{p}_{-i})$ is strictly decreasing in $\nu \in [0, \nu^*)$. Lastly, fix $\nu \in (\nu^*, \infty)$. By the definition of ν^*, we have $\mathcal{T}_i(\nu, \boldsymbol{p}_{-i}) = \varnothing$, in which case, by the definition of $G_i(\nu, \boldsymbol{p}_{-i})$, we obtain $G_i(\nu, \boldsymbol{p}_{-i}) = -2\, c_i$. Since $G_i(\nu, \boldsymbol{p}_{-i}) = -2\, c_i$ for all

$\nu \in (\nu^*, \infty)$ and $G_i(\nu, \boldsymbol{p}_{-i})$ is continuous in $\nu \in (0, \infty)$, it must be the case that $G_i(\nu^*, \boldsymbol{p}_{-i}) = -2\, c_i$ as well. Therefore, we have $G_i(\nu, \boldsymbol{p}_{-i}) = -2\, c_i$ for all $\nu \in [\nu^*, \infty)$.

Third, we show Part c. Assume that $G_i(0, \boldsymbol{p}_{-i}) > 0$. Since $\alpha_i^t > 0$ and $\beta_i^t > 0$, we have $\mathcal{T}_i(0, \boldsymbol{p}_{-i}) = T$ by the definition of $\mathcal{T}_i(\nu, \boldsymbol{p}_{-i})$. In this case, by the definition of $G_i(\nu, \boldsymbol{p}_{-i})$, we get $G_i(0, \boldsymbol{p}_{-i}) = \sum_{t \in T} (\alpha_i^t + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t) - 2\, c_i > 0$. Similarly, since $\alpha_i^t > 0$ and $\beta_i^t > 0$, we have $\mathcal{T}_i(\epsilon, \boldsymbol{p}_{-i}) = T$ for small enough $\epsilon > 0$. In this case, by the definition of $G_i(\nu, \boldsymbol{p}_{-i})$, we have $G_i(\epsilon, \boldsymbol{p}_{-i}) = \sum_{t \in T} (\alpha_i^t - \beta_i^t\, \epsilon + \sum_{j \neq i} \gamma_{i,j}^t\, p_j^t) - 2\, c_i$. Therefore, we have $\lim_{\epsilon \to 0} G_i(\epsilon, \boldsymbol{p}_{-i}) = G_i(0, \boldsymbol{p}_{-i})$, indicating that $G_i(\nu, \boldsymbol{p}_{-i})$ is continuous at $\nu = 0$. Noting Part a, it follows that $G_i(\nu, \boldsymbol{p}_{-i})$ is continuous in $\nu \in [0, \infty)$. Since $G_i(\nu, \boldsymbol{p}_{-i})$ is strictly decreasing in $\nu \in [0, \nu^*)$ and $G_i(v, \boldsymbol{p}_{-i}) < 0$ for all $\nu \in [\nu^*, \infty)$ by Part b and $G_i(\nu, \boldsymbol{p}_{-i})$ is continuous in $\nu \in [0, \infty)$ with $G_i(0, \boldsymbol{p}_{-i}) > 0$, there exists unique $\hat{\nu}$ such that $G_i(\hat{\nu}, \boldsymbol{p}_{-i}) = 0$. Next, assume that $G_i(0, \boldsymbol{p}_{-i}) = 0$. Clearly $\hat{\nu} = 0$ satisfies $G_i(\hat{\nu}, \boldsymbol{p}_{-i}) = 0$. Also, since $G_i(\nu, \boldsymbol{p}_{-i})$ is strictly decreasing in $\nu \in [0, \nu^*)$ and constant at a negative value for $\nu \in [\nu^*, \infty)$ by Part b, there cannot be another $\hat{\nu}$ such that $G_i(\hat{\nu}, \boldsymbol{p}_{-i}) = 0$. $\square$

In the next lemma, we show that if we fix the price trajectories of the firms other than firm i, then the best response of firm i, when viewed as a function of its initial inventory, is Lipschitz.

Lemma 8 *Fix the prices $\boldsymbol{p}_{-i}$ charged by the firms other than firm i and let $\{p_i^t(c_i) : t \in T\}$ be the optimal solution to problem* (1) *as a function of the initial inventory of firm i. Letting $\beta_{\min} = \min_{i \in N,\ t \in T} \beta_i^t$, for any two initial inventory levels $\hat{c}_i$ and $\tilde{c}_i$, we have*

$$\max_{t \in T} \left\{ |p_i^t(\hat{c}_i) - p_i^t(\tilde{c}_i)| \right\} \leq \frac{1}{\beta_{\min}}\, |\hat{c}_i - \tilde{c}_i|.$$

Proof. Since the prices $\boldsymbol{p}_{-i}$ charged by the firms other than firm i are fixed and we work with two different initial inventory levels, we drop the argument $\boldsymbol{p}_{-i}$ from $G_i(\nu, \boldsymbol{p}_{-i})$ and make the dependence of $G_i(\nu, \boldsymbol{p}_{-i})$ on c_i explicit. Thus, we use $G_i(\nu, c_i)$ to denote $G_i(\nu, \boldsymbol{p}_{-i})$ throughout the proof. For notational brevity, we let $\hat{p}_i^t = p_i^t(\hat{c}_i)$ and $\tilde{p}_i^t = p_i^t(\tilde{c}_i)$. Noting Lemma 2, we let $\hat{v}_i$ and $\tilde{v}_i$ be such that $G_i(\hat{v}_i, \hat{c}_i) = 0$ and $G_i(\tilde{v}_i, \tilde{c}_i) = 0$. Without loss of generality, we assume that $\hat{v}_i \geq \tilde{v}_i$. Since $G_i(\nu, c_i)$ is non-increasing in $\nu \in [0, \infty)$ by Lemma 7 and $G_i(\tilde{v}_i, \tilde{c}_i) = 0$, we have $G_i(\hat{v}_i, \tilde{c}_i) \leq 0$. Repeating the same argument in the first paragraph of the proof of Theorem 3, we also get $|\hat{p}_i^t - \tilde{p}_i^t| \leq \frac{1}{2}(\hat{v}_i - \tilde{v}_i)$ for all $t \in T$. The only difference is that we have $M_i = 0$ in this context since the prices charged by the firms other than firm i are fixed. If $\hat{v}_i \leq 2\, |\hat{c}_i - \tilde{c}_i| / \beta_{\min}$, then the last inequality implies that $|\hat{p}_i^t - \tilde{p}_i^t| \leq |\hat{c}_i - \tilde{c}_i| / \beta_{\min}$, which is the result that we want to show! In the rest of the proof, we proceed under the assumption that $\hat{v}_i > 2\, |\hat{c}_i - \tilde{c}_i| / \beta_{\min}$.

Noting that $G_i(\hat{v}_i, \hat{c}_i) = 0$ and $G_i(\nu, \hat{c}_i) < 0$ for all $\nu \in [\nu^*, \infty)$ by Lemma 7, the definition of ν^* implies that $\mathcal{T}_i(\hat{v}_i, \boldsymbol{p}_{-i}) \neq \varnothing$. If, otherwise, $\mathcal{T}_i(\hat{v}_i, \boldsymbol{p}_{-i}) = \varnothing$, then we obtain $\nu^* \leq \hat{v}_i$ by the definition of ν^*, which contradicts the fact that $G_i(\hat{v}_i, \hat{c}_i) = 0$, $G_i(\nu^*, \hat{c}_i) < 0$ and $G_i(\cdot, \hat{c}_i)$ is decreasing. Also, since $\hat{v}_i > 2\, |\hat{c}_i - \tilde{c}_i| / \beta_{\min} \geq 0$, by the definition of $G_i(\nu, c_i)$, we obtain $G_i(\hat{v}_i, \tilde{c}_i) - G_i(\hat{v}_i, \hat{c}_i) = -2\, (\tilde{c}_i - \hat{c}_i)$. Noting that $G_i(\hat{v}_i, \hat{c}_i) = 0$, the last equality yields $G_i(\hat{v}_i, \tilde{c}_i) = 2\, (\tilde{c}_i - \hat{c}_i)$. In the proof of Part a of Lemma 7, we show that $G_i(\nu, c_i) - G_i(\nu - \epsilon, c_i) \le -\sum_{t \in \mathcal{T}_i(\nu, \boldsymbol{p}_{-i})} \beta_i^t \, \epsilon$ for all $\nu \in (0, \infty)$ and $\epsilon > 0$ small enough that $\nu - \epsilon > 0$. Using this inequality with $\nu = \hat{v}_i$ and $c_i = \tilde{c}_i$, we obtain $G_i(\hat{v}_i - \epsilon, \tilde{c}_i) \ge G_i(\hat{v}_i, \tilde{c}_i) + \sum_{t \in \mathcal{T}_i(\hat{v}_i, \boldsymbol{p}_{-i})} \beta_i^t \, \epsilon$. Using the last inequality with $\epsilon = 2\, |\hat{c}_i - \tilde{c}_i| / \beta_{\min}$, since $\mathcal{T}_i(\hat{v}_i, \boldsymbol{p}_{-i}) \neq \varnothing$ and $G_i(\hat{v}_i, \tilde{c}_i) = 2\,(\tilde{c}_i - \hat{c}_i)$, we get

$$G_i\Big(\hat{v}_i - \frac{2}{\beta_{\min}}\, |\hat{c}_i - \tilde{c}_i|, \tilde{c}_i\Big) \ge 2\,(\tilde{c}_i - \hat{c}_i) + \beta_{\min} \frac{2}{\beta_{\min}}\, |\hat{c}_i - \tilde{c}_i| \ge 0 = G_i(\tilde{v}_i, \tilde{c}_i).$$

Noting that $G_i(\nu, c_i)$ is strictly decreasing in $\nu \in [0, \nu^*)$ and constant at a negative value for $\nu \in [\nu^*, \infty)$ by Part b of Lemma 7, having $G_i(\hat{v}_i - \frac{2}{\beta_{\min}}\, |\hat{c}_i - \tilde{c}_i|, \tilde{c}_i) \ge 0 = G_i(\tilde{v}_i, \tilde{c}_i)$ implies that $\hat{v}_i - \frac{2}{\beta_{\min}}\, |\hat{c}_i - \tilde{c}_i| \le \tilde{v}_i$. So, we obtain $|\hat{p}_i^t - \tilde{p}_i^t| \le \frac{1}{2}(\hat{v}_i - \tilde{v}_i) \le \frac{1}{\beta_{\min}} |\hat{c}_i - \tilde{c}_i|$ for all $t \in T$. $\square$