Delayed Purchase Options in Single-Leg Revenue Management

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Abstract: Many airline reservation systems offer the commitment option to their potential passengers. This option allows passengers to reserve a seat for a fixed duration before making a final purchase decision. In this study, we develop single-leg revenue management models that consider such contingent commitment decisions. We start with a dynamic programming model of this problem. This model is computationally intractable as it requires storing a multi-dimensional state space due to book-keeping of the committed seats. To alleviate this difficulty, we propose an alternate dynamic programming formulation that uses an approximate model of how the contingent commitments behave and we show how to extract a capacity allocation policy from the approximate dynamic programming formulation. In addition, we present a deterministic linear programming model that gives an upper bound on the optimal expected revenue from the intractable dynamic programming model. As the problem size becomes large in terms of flight capacity and the expected number of arrivals, we demonstrate an asymptotic lower bound for the deterministic linear programming model. Our extensive numerical study indicates that offering commitment options can noticeably increase the expected revenue even though offering a contingent commitment option may not always be in the best interest of the airline. Also, our results show that the proposed approximate dynamic programming model coordinates capacity allocation and commitment decisions quite well.

Keywords: Revenue management; airline; contingent commitment option; dynamic programming

1. Introduction. One of the main concerns in airline revenue management is to aid the decision makers to come up with strategies to increase the revenue. To this end, the control of the flight capacities plays an important role in most of these strategies. Capacity control is the practice of allocating seats to different fare classes to maximize the total expected revenue. Recently, the airline reservation systems started offering contingent commitment options that allow passengers to reserve a seat for a certain duration of time within the reservation period before making a buy or a leave decision. Since a commitment option allows passengers to keep a seat at a small fee, it has the potential to attract price sensitive customers as well as improve overall capacity utilization. However, it also creates another source of uncertainty leading to probable revenue loss due to empty seats.

As an example of a contingent commitment, consider a flight for which the airline offers the commitment option for all fare classes. Customers can still buy seats as usual. However, if a customer prefers to reserve a seat instead of buying it, then she can commit to a seat for a fixed non-refundable fee. Such a passenger would then be guaranteed a seat of the fare class until the end of a predetermined commitment period. The length of the commitment period is fixed by the airline. If the customer decides to purchase her committed ticket within this period, then she pays the ticket fare at the time of initial inquiry. Otherwise, she leaves the system without any reimbursement. In short, this option allows a passenger to delay her purchase decision with seat and price guarantee for the length of the commitment period.

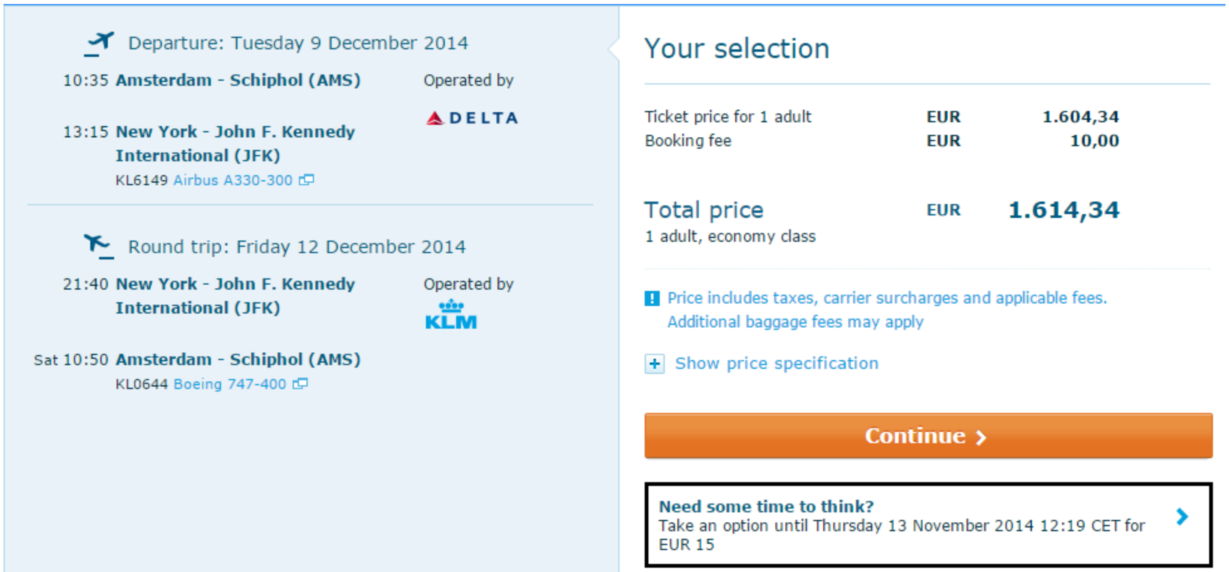


Figure 1: A screen shot of a contingent commitment option (KLM Website, 2014)

In practice, there are variants of the contingent commitment option. While some airline companies offer this option to all customers before they choose their flights, some other companies present this option right before customers purchase their tickets. Figure 1 shows a typical screen shot from an airline reservation website where a contingent commitment option is offered until a predetermined expiration date. We accessed the website in November and the flight departs in December. We observe that the website offers a single contingent commitment option and its fee is fixed. Although the commitment option resembles a typical travel insurance, there are two important differences. First, the commitment option holds the reservation for a fixed period of time, whereas the travel insurance is valid until the departure date. Second, contingent commitments allow passengers to exercise their options within the commitment period. However, a travel insurance allows free cancellation only if specific circumstances, like emergencies, arise. Therefore, a passenger is more likely to leave without exercising her contingent commitment option than to cancel her travel insurance.

From an airline perspective, every committed seat provides an additional revenue due to the non-refundable fee. However, reserving a seat, especially early in the reservation period, may result in rejecting a high fare class request at a later time, which in turn can lead to significant revenue losses. Therefore, the contingent commitment and the capacity control decisions should be simultaneously taken into consideration. To simplify the discussion, we refer to immediately purchased seats as *bookings* in the subsequent part.

In this paper, we address the joint problem of capacity allocation and commitment option for a single flight leg. We focus on contingent commitment options that are offered for fixed commitment fees and predefined expiration dates (as in Figure 1). We examine how offering these commitment options to customers affects overall revenue. Our problem setting is based on two independent streams of events; arrivals of booking and commitment requests and exercising the commitment option. At each time period, either a commitment request or a booking request can be realized independently. We need to decide either to accept or reject each arriving request. In other words, our policy determines whether to keep an available seat for a particular class of customers at that time period or not. We first introduce a dynamic programming formulation for this problem. However, this formulation requires keeping track of the remaining commitment time of each accepted contingent commitment request, and hence, makes use of a high-dimensional state variable. Consequently the dynamic program becomes intractable. As a remedy, we propose an approximation to the dynamic programming formulation that performs remarkably well as we demonstrate in our computational study. In addition to this approximation, we present deterministic linear programming models which provide upper bounds on value functions of the intractable dynamic programming formulation.

To the best of our knowledge, the concept of contingent commitment options has not been previously studied in the literature. Since the decision to leave without exercising a commitment option resembles a cancellation, our work is related to those works on single-leg capacity allocation with overbooking. The most relevant studies in the single-leg setting are given by Subramanian et al. (1999) and Aydın et al. (2013). Both of these studies propose a dynamic programming model for the capacity allocation problem with overbooking. Similar to our model, these studies allow cancellations but they consider only booking requests with no contingent commitments. Subramanian et al. (1999) consider cancellation and booking requests as a combined stream and assume that at most one of these events can occur at any discrete time epoch. On the other hand, Aydın et al. (2013) model the problem in a different way by allowing the arrival and cancellation processes to be independent. The main departing point of our work is that we consider two types of products, standard bookings and contingent commitments that can be sold by the airline. Once the airline accepts a contingent commitment request from a customer, it receives a fixed non-refundable fee and a seat is reserved for this customer for a certain duration. At the end of this duration, the customer can either purchase the ticket or leave without making use of the contingent commitment option. In addition, the dimension of the model that we are dealing with here is higher than that of the model given for the overbooking problem. The number of dimensions of the state variable in our model is equal to the length of the allowed commitment period, whereas the number of dimensions in overbooking and cancellation models is equal to the number of fare classes. Generally, the length of the allowed commitment period is larger than the number of fare classes.

Although we use airline reservation systems as the primary application area of this research, the commitment option as we consider here is applicable to any industry selling fixed, perishable capacity, such as; cargo, hotel and car rental. To make our point clear, we note that hotel reservation systems and car rental agencies are already exercising some options similar to the commitment option here. Car rental and hotel reservation systems offer both flexible and non-refundable products. While flexible products can be canceled without any penalty, non-refundable products are offered with various penalty options like charging the first day or the entire trip. For the non-refundable products the reservation systems present insurance policies for a fixed price. These insurance policies guarantee the refund of the whole reservation price, if the reservation is canceled. We will revisit these cases after presenting our models. Although, some of the problems in hotel and car rental industries are network based problems, the methods proposed in this paper may also be applied in these applications since in practice single-leg decomposition methods are frequently applied to network problems.

We make the following research contributions in this paper: (i) We develop a dynamic programming model to make the capacity allocation and contingent commitment decisions over a single flight leg. Due to the curse of dimensionality, we propose an alternate tractable dynamic programming model that approximates the actual contingent commitments process. (ii) We introduce deterministic linear programming approximations that give upper bounds on the intractable dynamic programming model. A lower bound is also obtained when the problem size becomes large in terms of capacities and the expected number of arrivals. (iii) Through computational experiments, we analyze the effects of offering a contingent commitment option. We demonstrate that under certain conditions, offering this option will increase the expected revenue of the flight even though offering the contingent commitment options is not always in the best interest of the airline. We also show that our approximate dynamic programming model performs remarkably well.

2. Review of Related Literature. There is an extensive literature on capacity allocation problems in revenue management. For a comprehensive review of this area, we refer reader to Phillips (2005), Talluri and van Ryzin (2004) and McGill and van Ryzin (1999). In the subsequent part of this section, we describe extensions to capacity allocation problems like cancellations and flexible products. In addition, we also discuss different options in pricing analysis and finance that are related to our work.

To compensate the revenue loss due to cancellations and no-shows, overbooking models have been studied. Early overbooking studies mainly focus on static models. Beckman (1958), Thompson (1961), and Coughlan (1999) develop static single leg capacity allocation and overbooking models by assuming the demand requests are static random variables. Several researchers have concentrated on dynamic overbooking models by considering the temporal dynamics of the demand process. Rothstein (1971) and Chatwin (1998) present a single fare class dynamic programming model to formulate the cancellations and the overbookings. Subramanian et al. (1999) study a more general setting than Chatwin (1998) by extending the overbooking problem to a multi-class problem. They point out the computational difficulties of the dynamic programming formulation and propose an approximation strategy.

Later studies concentrate on mitigating the effects of demand uncertainty. Karaesmen and van Ryzin (2004) formulate a two-stage overbooking model for multiple flight legs which allows substitution between flight legs in case of overbooking. In the first stage of the model, the airline takes reservations. In the second stage, overbooked passengers are bumped to substitute flights. Shumsky and Zhang (2009) analyze a dynamic upgrading model with fixed prices. Similar to work of Karaesmen and van Ryzin (2004), their optimal policy separates the allocation decision from the upgrade decision. On the other hand, Gallego and Stefanescu (2009) investigate the deterministic upgrading model where prices are fixed and flexible. Different than the previous models, they allow upgrading to any higher quality fare class.

Recently callable and flexible products have been introduced in the airline industry. Callable products give airlines the flexibility of accepting expensive fare class customers instead of low fare class customers. A buyer of such a product can be transferred to a later flight if there is no capacity left in the flight she has booked. In that case, the airline pays a pre-specified recall price to the customer (Gallego et al. (2008), Gallego et al. (2006)). Similarly, in flexible products the airline is free to assign the buyer to any of the pre-specified alternatives (Gallego and Phillips (2004)). Unlike the callable product, a flexible product guarantees a seat in those alternatives. Callable and flexible products appeal to customers who have low product valuation and flexible travel time. Gallego and Phillips (2004) show that offering the flexible product significantly increases the profitability. These options are also examined in the marketing science literature. Fay and Xie (2008) work on the concept of probabilistic goods. In their study, a probabilistic good corresponds to a set of multiple services that a buyer obtains with a probability. The probabilistic selling denotes the selling strategy where probabilistic goods and standard products are sold together. They examine the benefits of offering probabilistic goods. Similar to the flexible products, the opaque selling option is introduced in the travel industry. In opaque selling, product alternatives are concealed from a customer and she is unaware of the product she buys until the purchase. Anderson and Xie (2013) present a recent study on the opaque selling option and examine the cases where opaque selling is offered with regular full information selling. They show that offering opaque selling with regular selling improves the customer segmentation, and hence, increases the revenues. Gallego and Stefanescu (2010) give a nice overview of different options introduced in the service industries.

Lately, Sainam et al. (2009) investigate the benefits of call options in sport events. This option allows sport fans to reserve a ticket for the final game until the teams playing in the final are identified. If the option buyer decides to attend the game, she pays for the final. Otherwise, she cancels the ticket. Sainam et al. (2009) show that the call options provide extra revenue when they are offered with the advance purchase option. Balseiro et al. (2011) extend the work of Sainam et al. (2009) by including pricing analysis of call options. They propose a two-stage optimization model. In the first stage, a pricing problem is solved and in the second stage, given the fixed prices, the capacity allocation problem is solved. The problem is intractable. Therefore, they propose a deterministic approximation. Gallego and Sahin (2010) work on the partially refundable fares and show that offering partially refundable fares is more profitable than offering non-refundable and fully refundable fares. They propose an inter-temporal valuations model by considering both capacity provider and consumer. The commitment option that we discuss here can be considered as a special case of partially refundable fares where the passengers can get the refund, if they leave during the commitment period. However, these contingent commitment options bring an additional source of complexity as they can be utilized only within a certain time window.

Contingent commitments in our study are somewhat similar to the options in the finance literature. That literature focuses on pricing and exercise time of options. An option pricing problem can be modeled as a Markov decision problem. However, the resulting problem is hard to solve due to the curse of dimensionality. One approach is to use Monte Carlo simulation to generate good solutions (see for instance Board et al. (2003) for the pricing of European options). Another approach is to apply approximate dynamic programming to give lower and upper bounds on the value of the option (Longstaff and Schwartz, 2001; Tsitsiklis and Roy, 2001; Haugh and Kogan, 2008). The pricing problem is also approximated by solving linear programming models (Dempster and Hutton, 1999). We refer the reader to Trigeorgis (1996) for the review of pricing models. Several researchers work on the optimal time of exercising the real option. McDonald and Siegel (1986) work on the investment timing problem for an irreversible project and develop an investment rule when the value and the cost of the project are both stochastic. Rhys et al. (2002) use a first passage time approach to obtain expected waiting time to exercise an option. Han and Park (2008) develop a model to determine the exercise timing by considering the trade-off between early exercising and waiting.

3. Problem Formulation. We have a single flight leg with m fare classes and capacity C. The reservation horizon is partitioned into T time periods, and the flight departs at the beginning of period $T+1$. Figure 2 summarizes an arrival process in the booking horizon. At each time period, at most one customer arrives to the system with a particular fare class in mind. If this fare class is open for purchase, then the customer either purchases the ticket (Case 1) or decides to pay for the contingent commitment option. If the customer goes for the option, then she pays the option fee and is guaranteed a seat for the next s time periods. Right before the option expires, the customer decides whether or not to exercise the option and purchase the ticket. If the customer exercises the option, then she pays the airfare (Case 2). Otherwise, the option expires (Case 3).

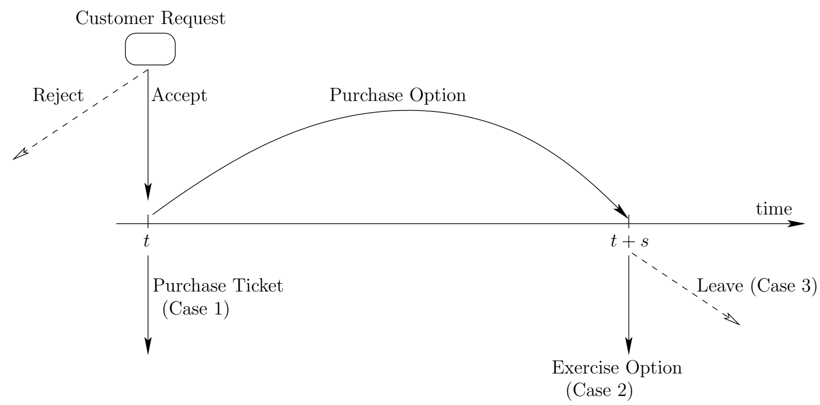


Figure 2: Dynamics following the arrival of a customer request at time t

Next, we formally define the problem and introduce our remaining notations. A customer that is interested in fare class i arrives at time period t with probability α_{it}. Then, she buys the commitment option with probability ν_i or books the seat with probability $(1-\nu_i)$. In other words, booking and commitment requests for fare class i arrive with probabilities $p_{it} = \alpha_{it}(1-\nu_i)$ and $q_{it} = \alpha_{it}\nu_i$, respectively. We assume that $\sum_{i=1}^{m}(p_{it}+q_{it}) \leq 1$ for all $t \in \{1,...,T\}$ and denote the probability of having no arrival by $p_{0t} = 1 - \sum_{i=1}^{m}(p_{it}+q_{it})$. After s time periods, she exercises the option and buys the seat with probability p_b or leaves the system with probability $p_l = 1 - p_b$.

As we pointed out in our preceding discussion, at each time period, we have to decide whether to accept or reject the arriving fare class request. When we accept a booking request for fare class i, then we generate a revenue of f_i. When we accept a commitment request for fare class i, we gain a fixed non-refundable revenue f^c at the period of request. After s periods, we generate a revenue of f_i with probability p_b, if the same customer decides to buy the ticket she had committed to. Talluri and van Ryzin (2004, Section 4.4.2) demonstrate that there is no difference in the total expected revenue if the accepted customer is charged at the time of reservation or later. Therefore, the expected revenue of an accepted commitment request for fare class i is $\phi_i := f^c + p_b f_i$. Each type of request consumes one capacity unit on the flight leg and the rejected requests or unexercised options simply leave the system.

We denote the total number of bookings and accepted commitments at a time period (decision epoch) t by x_t. To store the accepted contingent commitments between time periods $t-s$ and t, we designate an s-dimensional binary vector, $\mathbf{z}_t$. If there is an accepted commitment in one of the periods $\{t-s, t-s+1, \cdots, t-1\}$, then the corresponding component of $\mathbf{z}_t$ equals to 1; otherwise, it is set to 0. The pair x_t and $\mathbf{z}_t$ represents the state in our dynamic programming model of the problem. Note that the first element of $\mathbf{z}_t$ shows if there is a commitment request by a customer s time periods ago. At each time period, we need to check if there is such a customer and determine whether she makes an actual purchase decision or not. Letting z_{1t} be the first element of $\mathbf{z}_t$, the leaving passenger without making an actual purchase decision is represented by a Bernoulli random variable $\mathbf{B}(z_{1t}, p_l)$ having a success probability of p_l. As we move from period t to $t+1$, the first element of $\mathbf{z}_t$ needs to be dropped, and $\mathbf{z}_{t+1}$ is constructed by appending a binary variable to the remaining $s-1$ elements of $\mathbf{z}_t$. To denote this shifting operation, we define $\Gamma : \{0,1\}^{s+1} \mapsto \{0,1\}^s$ given by

$$\Gamma(\mathbf{z}, \zeta) = [\mathbf{0} \ I_s] \begin{bmatrix} \mathbf{z} \\ \zeta \end{bmatrix},$$

where $\mathbf{0}$ is an s-dimensional column vector consisting of zeros, I_s is an $s \times s$ identity matrix, and $\zeta \in \{0,1\}$. Using now this notation, if we accept a commitment request at time $t+1$, then $\mathbf{z}_{t+1} = \Gamma(\mathbf{z}_t, 1)$; otherwise, $\mathbf{z}_{t+1} = \Gamma(\mathbf{z}_t, 0)$.

We capture the decisions at time period t by an m-dimensional binary vector $\mathbf{u}_t = [u_{1t}, u_{2t}, \cdots, u_{mt}]^\intercal$ where u_{it} takes value 1 if we accept an arriving reservation request for fare class i at time period t, and takes value 0 if we reject an arriving reservation request at time period t. Since our accept-reject decision depends on the available capacity, the set of feasible decisions at time period t is given by

$$\mathcal{U}_t(x_t) = \{\mathbf{u}_t \in \{0,1\}^m : x_t + u_{it} \leq C, \quad i = 1, 2, \cdots, m\}.$$

We are ready to formulate the problem as a dynamic program. Let $J_t(x_t, \mathbf{z}_t)$ denote the expected optimal revenue from t up to T given that at time period t, the total number of bookings and commitments is x_t and the commitment history for s periods is $\mathbf{z}_t$. By the independence of the arrival and the commitment processes and the dynamic programming optimality principle, we obtain for every $1 \leq x_t \leq C$, $\mathbf{z}_t \in \{0,1\}^s$ and $t = 1, 2, \cdots, s$ that

$$\begin{aligned} J_t(x_t, \mathbf{z}_t) = \max_{\mathbf{u}_t \in \mathcal{U}(x_t)} \Bigg\{ & \sum_{i=1}^{m} p_{it} \Big\{ f_i u_{it} + J_{t+1}(x_t + u_{it}, \Gamma(\mathbf{z}_t, 0)) \Big\} + \\ & \sum_{i=1}^{m} q_{it} \Big\{ \phi_i u_{it} + J_{t+1}(x_t + u_{it}, \Gamma(\mathbf{z}_t, u_{it})) \Big\} + p_{0t} J_{t+1}(x_t, \Gamma(\mathbf{z}_t, 0)) \Bigg\} \end{aligned} \tag{1a}$$

and for $s < t \leq T$,

$$\begin{aligned} J_t(x_t, \mathbf{z}_t) = \max_{\mathbf{u}_t \in \mathcal{U}(x_t)} \Bigg\{ & \sum_{i=1}^{m} p_{it} \Big\{ f_i u_{it} + \mathbb{E} J_{t+1}(x_t + u_{it} - \mathbf{B}(z_{1t}, p_l), \Gamma(\mathbf{z}_t, 0)) \Big\} + \\ & \sum_{i=1}^{m} q_{it} \Big\{ \phi_i u_{it} + \mathbb{E} J_{t+1}(x_t + u_{it} - \mathbf{B}(z_{1t}, p_l), \Gamma(\mathbf{z}_t, u_{it})) \Big\} + \\ & p_{0t} \mathbb{E} J_{t+1}(x_t - \mathbf{B}(z_{1t}, p_l), \Gamma(\mathbf{z}_t, 0)) \Bigg\}. \end{aligned} \tag{1b}$$

The boundary condition is simply $J_{T+1}(x_{T+1}, \mathbf{z}_{T+1}) = 0$. Since a contingent commitment makes the purchase decision at the end of the commitment period, we do not observe any commitment purchase decisions during the first s time periods. Since options expire at the end of commitment period, we need to compute the expectation of optimal value functions using a Bernoulli event after time period s. This means for $z_{1t} = 1$ that

$$\mathbb{E} J_{t+1}(x_t + u_{it} - \mathbf{B}(z_{1t}, p_l), \Gamma(\mathbf{z}_t, 0)) = p_b J_{t+1}(x_t + u_{it}, \Gamma(\mathbf{z}_t, 0)) + p_l J_{t+1}(x_t + u_{it} - 1, \Gamma(\mathbf{z}_t, 0)).$$

Clearly, $J_1(0, \mathbf{0})$ gives the optimal expected total revenue at the beginning of the planning horizon, where $\mathbf{0}$ represents the fact that we start with no commitments.

4. Approximate Model. We note that the state variable $\mathbf{z}_t$ in the dynamic model may involve many dimensions in actual applications. Thus, solving the recursive equation through standard dynamic programming tools can be computationally demanding. Therefore, we propose an approximation to our dynamic programming formulation. Our approximation hinges on the assumption that each commitment, independently of other commitments, can exercise, leave or retain with probabilities q_e, q_l, and q_r *at each time period* until the departure of the flight. We calibrate these probabilities so that the expected amount of time that a contingent commitment stays in the system is exactly s periods, and a contingent commitment results in a final purchasing decision with probability p_b, which is the probability that a customer with a commitment purchases the ticket in the original model. In other words, we choose q_r and q_e such that $1/(1 - q_r) = s$ and $p_b = q_e + q_e q_r + q_e q_r^2 + \dots$ Thus, each accepted commitment request *eventually* buys the ticket with probability p_b. Therefore, as in Section 3, $\phi_i = f^c + p_b f_i$ gives the expected revenue obtained from a fare class i commitment request. Observe that the proposed way of calculating p_b somewhat underestimates the value of q_e as this sum, at any time period t, should involve only $T - t$ terms and not infinite terms. However, if we use such a finite sum, then we have to use a time dependent q_e parameter, which disagrees with our approximation approach. Once we choose q_r and q_e in this fashion, we obtain $q_l = 1 - q_r - q_e$.

Furthermore given that there are y accepted commitments, the random numbers of exercised, $\mathbf{M_e}(y)$, not exercised, $\mathbf{M_l}(y)$ and retained, $\mathbf{M_r}(y)$ commitments in period t follow collectively the multinomial distribution with parameters q_e, q_l, q_r, and number of trials y. Note that under this probabilistic setting, a committed passenger may stay in the system until the departure time. We also assume that an accepted commitment request cannot make a buy or leave decision in the time period she is accepted. We believe that this assumption is more realistic since in practice the duration of a time period is quite short. An appealing feature of this modeling approach is that it avoids the necessity to keep track of how long each accepted contingent commitment has been in the system since a contingent commitment makes a decision to exercise, leaves or retain the option at each time period independently. In this case, the state variable in the dynamic programming formulation of the commitment problem collapses to two scalars; the number of bookings and the number of accepted contingent commitments.

Let x_t and y_t be the total number of reservations (including both contingent commitments and bookings) and contingent commitments at time period t, respectively. Then, the recursive equations for the proposed approximate dynamic programming model is given by

$$\begin{aligned} V_t(x_t, y_t) = \max_{\mathbf{u}_t \in \mathcal{U}(x_t)} \Bigg\{ & \sum_{i=1}^{m} p_{it} \Big\{ f_i u_{it} + \mathbb{E} V_{t+1}(x_t + u_{it} - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t)) \Big\} + \\ & \sum_{i=1}^{m} q_{it} \Big\{ \phi_i u_{it} + \mathbb{E} V_{t+1}(x_t + u_{it} - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t) + u_{it}) \Big\} + \\ & p_{0t} \mathbb{E} V_{t+1}(x_t - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t)) \Bigg\}. \end{aligned} \qquad (2)$$

Again, the boundary condition is simply $V_{T+1}(x_{T+1}, y_{T+1}) = 0$. In this formulation, $x_t - \mathbf{M_l}(y_t)$ and $\mathbf{M_r}(y_t)$ represent the remaining number of reservations and commitments, given the state of reservations at the beginning of time period t is (x_t, y_t) and we do not accept anybody during that time period. On the other hand, if we accept a commitment request at time period t, the state of the system becomes $(x_t + 1 - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t) + 1)$, since we assume that commitments cannot leave within the period they are accepted.

Note that in both models, the purchase probability of a contingent commitment is class independent. In case of the dynamic model (1a)-(1b), we could have relaxed this assumption and worked with class dependent purchase probabilities. Then, we would have needed to store the fare class of each accepted commitment, which would have required holding even a larger state space. In case of the approximate model (2), however, q_e and q_r values are class independent by definition. We could have used weighted averages to set both probabilities. In fact this was the approximation used by Aydın et al. (2013). If we had used such an approach, then we would have added one more level of approximation to our dynamic programming model. Therefore, we avoided this kind of construction and decided to work with a purchase probability that is class independent.

Before we discuss the optimal policy, let us note that the way we use the commitment option in the approximate dynamic model resembles similar options offered in the service industry. For instance, the insurance policies are also commonly offered to guarantee reservations. In this case, the customers can leave at any time until they receive the service. However, this default option in insurance policies is just an assumption in our approximate model.

The optimal policy of problem (2) can be summarized as follows: Given the state variables (x_t, y_t) at time period t, the optimal decisions at time period t are given by

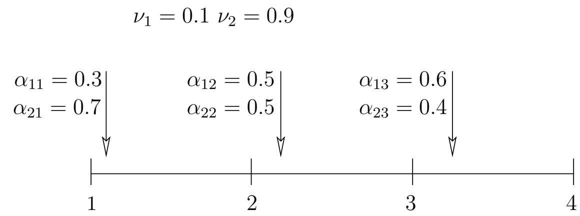
$$u_{it}^* = \begin{cases} 1, & \text{if} \quad (1-\nu_i)(f_i + V_{t+1}(x_t+1, y_t)) + \nu_i(\phi_i + V_{t+1}(x_t+1, y_t+1)) \geq V_{t+1}(x_t, y_t) \text{ and } x_t < C; \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Next, we present that optimal decisions have a nested structure under certain conditions. We defer the proof of the proposition to the appendix.

Proposition 4.1 *Suppose the probability of a request being a commitment is class independent; that is,* $\nu_1 = \nu_2 = \cdots = \nu_m$. *Then, given the fare ordering* $f_1 \geq f_2 \geq ... \geq f_m$*, and hence, the ordering of the expected commitment revenues,* $\phi_1 \geq \phi_2 \geq ... \geq \phi_m$*, we have* $u_{1t}^* \geq u_{2t}^* \geq ... \geq u_{mt}^*, t = 1, ..., T$.

The assumption in Proposition 4.1 seems crucial as we can give a simple counter example where the optimal policy does not have a nested structure. Figure 3 illustrates such an example. Given that there are $x = 1$ reservations and $y = 0$ commitments at the beginning of each time period, the optimal decisions are computed. The optimal policy table is given in the lower part of the figure. As this table shows, although a request for the low fare class is accepted, the expensive fare class request is rejected for the first two periods.

5. Deterministic Linear Program. An alternate approximation approach is to model a deterministic linear program (DLP) that corresponds to the dynamic programming model (1a)-(1b). In this problem, our decision variables are the number of accepted reservations for each fare class at each time period and the remaining capacity at the beginning of each time period. To formulate this linear program, let w_{it} be the number of the bookings and commitments that we plan to accept for fare class i at



u^*_{it}			
1^{st} fare class	0	0	1
2^{nd} fare class	1	1	1

Optimal Policy Table,(x,y)=(1,0)

Figure 3: A counter example when the assumption in Proposition 4.1 does not hold under class dependent commitment request probabilities ($m = 2$, $C = 25$, $f_1 = 26$, $f_2 = 25$, $f^c = 10$, $s = 2$, $p_b = 0.7$)

time period t. We also denote the remaining capacity at time period t by ϑ_t. Since an arriving customer either buys the commitment option with probability ν_i or books the seat with probability $(1 - \nu_i)$, the expected number of booked and committed fare class i seats at time period t are given by $(1-\nu_i)w_{it}$ and $\nu_i w_{it}$, respectively. Consequently, the total expected number of reservations accepted at time period t is $\sum_{i=1}^m w_{it}$. Then, the deterministic linear program has the following form:

$$
\begin{aligned}
\text{maximize} \quad & \sum_{t=1}^{T}\sum_{i=1}^{m} f_i(1-\nu_i)w_{it} + \sum_{t=1}^{T}\sum_{i=1}^{m} \phi_i \nu_i w_{it} && (4)\\
\text{subject to} \quad & \vartheta_1 = C, && (5)\\
& \vartheta_t = \vartheta_{t-1} - \sum_{i=1}^{m}(1-\nu_i)w_{i(t-1)} - \sum_{i=1}^{m}\nu_i w_{i(t-1)}, \qquad t = 2,\cdots,s+1, && (6)\\
& \vartheta_t = \vartheta_{t-1} - \sum_{i=1}^{m}(1-\nu_i)w_{i(t-1)} - \sum_{i=1}^{m}\nu_i w_{i(t-1)} + \sum_{i=1}^{m}\nu_i w_{i(t-s-1)}p_l, \quad t = s+2,\cdots,T, && (7)\\
& \vartheta_{T+1} = \vartheta_T - \sum_{i=1}^{m}(1-\nu_i)w_{iT} - \sum_{i=1}^{m}\nu_i w_{iT} + \sum_{k=T-s}^{T}\sum_{i=1}^{m}\nu_i w_{ik}p_l, && (8)\\
& \sum_{i=1}^{m} w_{it} \le \vartheta_t, \quad t = 1,...,T, && (9)\\
& w_{it} \le \alpha_{it}, \qquad i = 1,...,m; t = 1,...,T, && (10)\\
& \vartheta_t \ge 0, \qquad t = 1,...,T+1, && (11)\\
& w_{it} \ge 0, \qquad i = 1,...,m; t = 1,...,T. && (12)
\end{aligned}
$$

Constraints (5)-(8) keep track of the remaining capacity at each time period. Constraints (9) ensure that the accepted reservations at each time period do not exceed the available capacity at the beginning of that time period. Similarly, constraints (10) guarantee that the reservation requests that we plan to accept do not exceed the expected number of arrivals. Moreover, due to constraints (10), arriving requests can be accepted partially. Note that the nonnegativity constraint on ϑ_t prevents overbooking. By substituting constraints (5)-(7) into constraint (8) and rewriting ϑ_{T+1} in terms of ϑ_t, $t \in \{1,\cdots,T\}$, we obtain that $\vartheta_{T+1} = C - \sum_{t=1}^T\sum_{i=1}^m w_{it} + p_l\sum_{t=1}^T\sum_{i=1}^m \nu_i w_{it}$. Since $\vartheta_{T+1} \ge 0$, by rearranging the terms we observe that $\sum_{t=1}^T\sum_{i=1}^m(1-\nu_i)w_{it} + (1-p_l)\sum_{t=1}^T\sum_{i=1}^m \nu_i w_{it} \le C$. In addition, an accepted commitment customer does not exercise the option with probability p_l, and hence, the available capacity increases in this case. Therefore, the total expected number of assigned seats (bookings and option buyers) may exceed the capacity. In this model, this excess amount depends on the probability of not exercising. We denote the optimal objective function value of (4)-(12) by Z^*_{DLP}.

There are two uses of DLP. First, it gives a policy to accept or reject the product requests. Let $\{w^*_{it}, \forall i, t\}$ be the optimal value of the decision variables in problem (4)-(12). Then, according to the policy dictated by DLP, a booking or a commitment request is accepted with probability w^*_{it}/α_{it}. Second, its optimal objective value provides an upper bound on the maximum expected revenue over the whole planning horizon. This is an intuitive consequence of the linear programming approximation to a dynamic model. In fact, results similar to the next proposition widely appear in revenue management literature; see Talluri and van Ryzin (1998) and Gallego and van Ryzin (1997).

Proposition 5.1 *The optimal objective value of the DLP model gives an upper bound on the dynamic programming model* (1a)-(1b)*. That is,* $J_1(0, \mathbf{0}) \leq Z^*_{DLP}$.

The proof of Proposition 5.1 is given in the appendix. We can obtain a tighter upper bound by using a randomized linear program. This is a well-known result in the revenue management literature and it involves a somewhat standard analysis. Therefore, we omit this discussion and refer interested reader to Talluri and van Ryzin (1999) and Kunnumkal et al. (2011). Instead, we focus on obtaining an asymptotic lower bound. To obtain this bound, we make use of another upper bounding problem as we explain next.

Note that problem (4)-(12) ensures that the remaining capacity at each time period, ϑ_t, is non-negative. By relaxing this constraint, we can give an upper bound on the DLP model (4)-(12) as follows:

$$\begin{aligned}
\text{maximize} \quad & \sum_{t=1}^{T}\sum_{i=1}^{m} f_i(1-\nu_i)w_{it} + \sum_{t=1}^{T}\sum_{i=1}^{m}\phi_i\nu_i w_{it} && (13)\\
\text{subject to} \quad & \sum_{t=1}^{T}\sum_{i=1}^{m}(1-\nu_i)w_{it} + \sum_{t=1}^{T}\sum_{i=1}^{m} p_b\nu_i w_{it} \leq C, && (14)\\
& w_{it} \leq \alpha_{it}, \qquad i = 1,...,m; t = 1,...,T, && (15)\\
& w_{it} \geq 0, \qquad i = 1,...,m; t = 1,...,T. && (16)
\end{aligned}$$

In constraint (14), $p_b = 1 - p_l$ is the probability of exercising the option. We denote the optimal objective function of this model by Z^*_{DLP-UB}. Here, constraint (14) is obtained by substituting constraints (5)-(7) into constraint (8).

Remark 5.1 *When there is no commitment option ($s = 0$) or the probability of buying the committed seat equals to 1 ($p_b = 1$), DLP given by (4)-(12) boils down to the standard capacity allocation problem. Furthermore, when $Z^*_{DLP-UB} = Z^*_{DLP}$, the dual variables corresponding to constraints (5)-(8) in problem (4)-(12) are all equal. Therefore, the dual of problem (4)-(12) can be reduced to a one-dimensional unconstrained problem, and it can be solved very efficiently by any variant of the bisection method.*

Now we are ready to obtain an asymptotic lower bound on the distance between the optimal objective function value of DLP and the optimal expected revenue of the dynamic programming model (1a)-(1b).

Our analysis follows a similar approach as in Gallego et al. (2004). However, in our case, we need to consider the expiration of the option explicitly. To provide an asymptotic bound, we scale the capacity of the flight and the expected demand linearly with the same rate κ. We introduce a sequence of problems $\{\mathcal{P}^\kappa : \kappa \in \mathbb{Z}_+\}$ indexed by parameter κ. Problem $\mathcal{P}^\kappa$ has κT time periods in the planning horizon and the capacity of the flight is κC. Moreover, the arrival probabilities α_{it} at time periods $\{\kappa(t-1)+1, ..., \kappa t\}$ in problem $\mathcal{P}^\kappa$ are the same as the arrival probabilities at time period t in problem $\mathcal{P}^1$. Therefore, the probability of a reservation request for fare class i at time period t is given by $\alpha_{i\lceil t/\kappa \rceil}$, where operator $\lceil . \rceil$ rounds up the values passed to it. We note that the problem described in Section 3 is $\mathcal{P}^1$. The flight capacity in problem $\mathcal{P}^\kappa$ is κ times the capacity of the flight in problem P^1. Similarly, the length of the booking horizon in problem $\mathcal{P}^\kappa$ is κ times the length of the booking horizon in problem $\mathcal{P}^1$. Consequently, the expected total booking demand and the expected total commitment demand for fare class i in problem $\mathcal{P}^\kappa$ is

$$\sum_{t=1}^{\kappa T} \alpha_{i\lceil t/\kappa \rceil} = \kappa \sum_{t=1}^{T} \alpha_{it}.$$

This implies that the expected numbers of reservation requests in problem $\mathcal{P}^\kappa$ are κ times larger than those in problem $\mathcal{P}^1$. Consequently, problem $\mathcal{P}^\kappa$ is a scaled version of problem $\mathcal{P}^1$.

Our goal is to explore how the performance of the policy derived from the deterministic linear program changes as the capacity on the flight leg and the demand progressively get larger. We do not scale the length of the commitment horizon s, but the commitments do not lose their importance in our asymptotic scaling regime, since as we scale up the demand, the number of customers that purchase the commitment option also scales up.

We consider the linear programming model (13)-(16) for problem $\mathcal{P}^\kappa$. Let Z^κ_{DLP-UB} denote the optimal objective value of the upper bound on DLP for the scaled problem $\mathcal{P}^\kappa$. Likewise, Z^κ_{DLP} denotes the optimal objective value of the scaled deterministic linear program given by problem (4)-(12), and $J^\kappa_1(0, \mathbf{0})$ stands for the optimal expected total revenue for the scaled problem $\mathcal{P}^\kappa$ that we obtain by solving the corresponding dynamic program. Proposition 5.1 shows that the optimal objective value of the deterministic linear program provides an upper bound on the optimal expected total revenue. Thus, we have $Z^\kappa_{DLP} \geq J^\kappa_1(0, \mathbf{0})$. Since $Z^\kappa_{DLP-UB} \geq Z^\kappa_{DLP}$, we also have $Z^\kappa_{DLP-UB} \geq J^\kappa_1(0, \mathbf{0})$. The lower bound on these inequalities along with an asymptotic behavior are shown in the next proposition. We give the proof of this result in the appendix.

Proposition 5.2 *Given $\epsilon \in [1 - p_b, 1]$ and $\kappa > 0$, we have*

$$Z^\kappa_{DLP-UB} \geq Z^\kappa_{DLP} \geq J^\kappa_1(0, \mathbf{0}) \geq \left(1 - \epsilon - \frac{CV^2}{\kappa \epsilon^2}\right) Z^\kappa_{DLP-UB},$$

where CV denotes the maximum coefficient of variation over bookings and commitments for all fare classes (see the appendix). Therefore,

$$p_b \leq \lim_{\kappa \to \infty} \frac{J^\kappa_1(0, \mathbf{0})}{Z^\kappa_{DLP}} \leq 1.$$

The first part of this proposition gives a lower bound for the scaled problems of DLP and the dynamic programming model (1a)-(1b). The asymptotic result in the second part implies that the optimal objective function value of the DLP is at most $1/p_b$ multiple of the dynamic model as the problem size gets large in terms of the capacity and the expected demand. This limiting behavior also shows that DLP becomes asymptotically tight as p_b becomes closer to 1. This is expected, since the commitment problem becomes a standard capacity allocation problem when $p_b = 1$ (Talluri and van Ryzin, 1998).

6. Computational Experiments. In this section, we conduct simulation experiments to evaluate the effects of offering the contingent commitment option. We also provide a sensitivity analysis with respect to various parameters. Moreover, we compare the performance of our dynamic model against other benchmark strategies. We begin by describing the benchmark strategies.

Approximate Dynamic Model (ADM): This is the solution method that we develop in this paper. That is, we solve the dynamic program in (2) to obtain the optimal policy. Then, we use the decision rule (3) as our accept-reject policy for booking and commitment requests.

Standard Booking Strategy (SBS): This policy ignores the commitment requests and only accepts the standard booking requests. Therefore, the no arrival probability at time period t becomes $(1 - \sum_{i=1}^{m} p_{it})$ in this policy. The optimal booking policy is then determined by solving the problem as a standard capacity allocation problem (Talluri and van Ryzin, 2004, Section 2.5).

Deterministic Linear Program (DLP): This is the solution method described in Section 5. We solve the problem (4)-(12) to obtain the optimal values of the variables w_{it}^*. Provided that there is sufficient remaining capacity, we accept a reservation request for fare class i with probability w_{it}^*/α_{it} at time period t.

In the sequel, we refer to the average revenue obtained by the optimal policy of the dynamic model given by (1a)-(1b) as **DM**. Recall that this model is computationally intractable for long commitment periods. Hence, we test the models with respect to DM for only small instances. We simulate the arrival of reservation requests and option decisions over discrete time periods $\{1, ..., T\}$. At each time period, we first generate an arrival request and then apply the corresponding policy. While an accepted booking request for fare class i generates a revenue of f_i, an accepted commitment request generates a revenue of f^c. After the arrival process, we check whether there is a commitment made s periods ago and simulate a purchase or leave decision. Each commitment passenger in fare class i buys the ticket with probability p_b generating an additional revenue of f_i, or leaves the system.

To test the performances of the booking policies against varying arrival intensities, we use the load factor parameter ρ. Noting that the total expected demand for the flight is $\sum_{t=1}^{T} \sum_{i=1}^{m} (p_{it} + p_b q_{it})$, the load factor is given by

$$\rho = \frac{\sum_{t=1}^{T} \sum_{i=1}^{m} (p_{it} + p_b q_{it})}{C}.$$

The way we generate arrival probabilities is quite similar to the one given by Aydın et al. (2013). We assume that the lower fare class requests arrive more frequently than the higher fare classes in early periods. In all our numerical experiments, we set the capacity of the plane, the length of the planning horizon and the number of fare classes to $C = 100$, $T = 300$ and $m = 4$, respectively. The fares are evenly distributed between 250 and $1,000$.

6.1 Benchmarking Study. Our experimental design is based on various factors of the load factor (ρ), the commitment period (s), the commitment fee (f^c), the probability of buying the committed seat (p_b), and the splitting probability of commitment arrivals (ν). We use load factor values $\rho \in \{1.2, 1.6\}$ corresponding to low and high loads. We select the commitment period lengths from the set $\{5, 25, 50\}$ to represent short, medium and long commitment intervals. The commitment fees $f^c \in \{40, 80\}$ are used to represent low and high fees. We also test the models for varying buy probabilities $p_b \in \{0.4, 0.7\}$. The last parameter set comes from the splitting probability of contingent commitments (ν_i values). We give two sets of values to represent low and high commitment arrivals. These are $\nu_L := (0.10, 0.15, 0.20, 0.25)$ and $\nu_H := (0.40, 0.45, 0.50, 0.55)$ where the values in each set are ordered from expensive to cheap fare class. We label our test problems by using all combinations of these parameters.

As mentioned in Section 5, DLP provides an upper bound on the maximum total expected revenue obtained by the dynamic model over the time periods $\{1, ..., T\}$. Moreover, we also show that the optimal objective function value of DLP is at most $1/p_b$ multiple of the dynamic model as the problem size gets large in terms of the capacity and the expected demand. Table 1 shows the optimal expected revenues obtained by DM ($J_1(0, \mathbf{0})$) and DLP (Z^*_{DLP}) for different test instances. The first four columns indicate the characteristics of the test instances. The next two columns give the optimal objective values of DM and DLP, respectively. The last column gives the percentage gaps with respect to Z^*_{DLP}. We compare the expected revenues for $s = 5$. The results show that the upper bound provided by DLP is within 1.5% of DM. In particular, for the test instances with high load factor and high commitment demand, this percentage gap drops down. We observe that the quality of the upper bound seems to be mostly affected by the tightness of the flight capacity.

Table 1: Optimal expected revenues ($s = 5$)

Instances						% Gap with DM
ρ	$\nu_\bullet$	f^c	p_b	DM	DLP	DLP
1.2	ν_H	40	0.4	65,657	66,420	1.16%
		40	0.7	64,390	65,185	1.23%
		80	0.4	68,251	69,045	1.16%
		80	0.7	66,563	67,391	1.24%
	ν_L	40	0.4	63,547	64,339	1.25%
		40	0.7	62,899	63,687	1.25%
		80	0.4	64,308	65,116	1.26%
		80	0.7	63,622	64,424	1.26%
1.6	ν_H	40	0.4	75,598	76,113	0.68%
		40	0.7	74,469	75,101	0.85%
		80	0.4	78,072	78,583	0.65%
		80	0.7	76,563	77,198	0.83%
	ν_L	40	0.4	73,582	74,214	0.86%
		40	0.7	73,002	73,664	0.91%
		80	0.4	74,259	74,890	0.85%
		80	0.7	73,649	74,310	0.90%

Next, we test the performances of the models against the dynamic model given by (1a)-(1b). We estimate the net expected revenues by simulating the arrivals of booking and commitment requests over 5,000 sample paths. We set the length of the commitment period to 5 for the instances where we compare our models with respect to DM. Table 2 shows the average total revenues and percentage gaps between DM and the remaining solution methods. The first four columns in Table 2 show the characteristics of the test instances. The next four columns give the expected total revenues obtained by DM, ADM, SBS, and DLP, respectively. The last three columns give the percentage gaps between DM and the remaining solution methods. Comparing the percentage gaps under this setup, we observe that the performances of DM and ADM are very close, especially for high values of buy probabilities p_b. In the worst-case, which corresponds to the value of $p_b = 0.4$, the gap between ADM and DM is less than 0.1%. Moreover as the load factor increases, the percentage gap between DM and ADM decreases. When the arrival intensity is high, models can compensate the revenue loss due to empty seats. On the other hand, we observe that there is a noticeable performance gap between ADM and SBS. The performance of SBS improves slightly when the load factor is high and splitting probability is low. However even in this case, it performs worse than ADM. A noteworthy observation is the relatively large difference between ADM and SBS even when the load factor is high ($\rho = 1.6$) and the splitting probability is low (ν_L). Because in this case there is ample booking requests to use the full capacity of the flight.

Table 2: Average total revenues over 5,000 runs ($s = 5$)

Instances								% Gap with DM		
ρ	$\nu_\bullet$	f^c	p_b	DM	ADM	SBS	DLP	ADM	SBS	DLP
1.2	ν_H	40	0.4	65,748	65,732	51,455	64,062	0.024%	21.738%	2.564%
		40	0.7	64,443	64,437	42,685	62,684	0.010%	33.764%	2.730%
		80	0.4	68,298	68,294	51,455	66,540	0.007%	24.660%	2.574%
		80	0.7	66,459	66,453	42,685	64,648	0.009%	35.772%	2.725%
	ν_L	40	0.4	63,543	63,541	60,362	61,872	0.003%	5.007%	2.630%
		40	0.7	62,883	62,883	58,056	61,230	0.001%	7.677%	2.629%
		80	0.4	64,313	64,308	60,362	62,602	0.008%	6.143%	2.660%
		80	0.7	63,595	63,595	58,056	61,922	0.000%	8.710%	2.631%
1.6	ν_H	40	0.4	75,579	75,569	63,738	73,508	0.014%	15.667%	2.740%
		40	0.7	74,435	74,434	56,267	72,257	0.002%	24.409%	2.926%
		80	0.4	78,049	78,043	63,738	75,891	0.009%	18.336%	2.765%
		80	0.7	76,543	76,541	56,267	74,315	0.003%	26.490%	2.911%
	ν_L	40	0.4	73,547	73,545	71,411	71,505	0.003%	2.905%	2.776%
		40	0.7	72,964	72,963	69,289	70,972	0.001%	5.036%	2.730%
		80	0.4	74,221	74,220	71,411	72,162	0.001%	3.787%	2.774%
		80	0.7	73,607	73,606	69,289	71,597	0.002%	5.867%	2.731%

Next, we report our results for larger values of the commitment period, s. For comparison, we also present the results obtained when s is 5. Table 3 presents the performances of the benchmark strategies with respect to various test instances. The columns have the same interpretation as in Table 2. To emphasize the effect of the commitment decision in these experiments, we fix the commitment fee to the highest value, $f^c = 80$. As depicted in Table 3, the total expected revenues decrease as the length of the commitment period increases since the revenues obtained from the contingent commitments decrease with the length of the commitment period. However, this loss can be compensated with the later arrivals of the booking requests. Thus, the decrease in revenue is more striking when the arrival intensity is low ($\rho = 1.2$). On the other hand, the results indicate that ADM consistently provides the highest total expected revenues. However as the length of the commitment period increases, the percentage gaps between ADM and the other solution methods decrease. This behavior can be attributed to the impact of the commitment period on retain, buy and leave probabilities. Recall that $q_r = (s-1)/s$; so, q_r increases as the length of the commitment period increases. Therefore, when s is high, our proposed dynamic model presumes that each accepted commitment request waits until the departure time for purchasing or cancelling the option (over-estimates the commitment period). Consequently, it may fail to capture the actual dynamics of the system and its performance deteriorates. Moreover, as the length of the commitment period increases, there exist instances where the performances of SBS and DLP are somewhat close.

The results indicate that offering a contingent commitment option is most beneficial when the purchase probability is high and the length of the commitment period is short. When we compare the performances of ADM and SBS, we observe that offering the commitment option brings in a significant revenue increase, even if the customer arrival intensity for this option is low.

6.2 When to Offer the Commitment Option. In this part, we investigate the effects of the contingent commitment option. We set the load factor (ρ) to 1.6, the length of the commitment period (s) to 50, and the commitment fee (f^c) to 80. The splitting probabilities of commitments for all fare classes are set to the same value of 0.5. Initially, we study the potential revenue improvements of offering the commitment option relative to offering only standard bookings. Since the dynamic model (1a)-(1b) is computationally intractable for long commitment periods, we make an analysis on the approximate model (2). To measure the effect of commitment option, we generate two models which accept the contingent commitment requests during a limited time period. While the first model allows commitment arrivals only in the first τ periods, the second model allows them only in the last τ. These models are denoted by FCM and LCM, respectively. The time period of length τ during which any commitment requests may be processed is called commitment interval. For instance, $\tau = 10$ means that FCM allows commitments during only the first 10 periods and LCM allows them only in the last 10 periods. On the other hand, when $\tau = T$, both FCM and LCM boil down to the ADM model where commitment arrivals are allowed during the whole reservation horizon.

Figure 4 shows the optimal objective values of these models with respect to different purchase probabilities. In this figure, the horizontal axis represents the commitment interval τ. As Figure 4 shows, FCM performs better than LCM when the probability of purchase is low. Due to a high retain probability (q_r) and low purchase probability (p_b), offering the commitment option later in the reservation horizon may result in empty seats. Since FCM accepts contingent commitment requests early in the reservation period, it can compensate the empty seats resulting from not exercised commitments with the late booking arrivals.

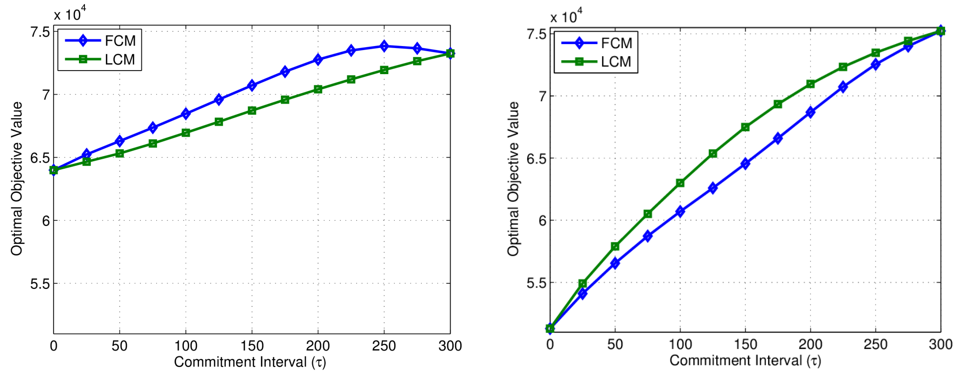
On the other hand, as the purchase probability increases, the performance of LCM improves. Since

Table 3: Average total revenues over 5,000 runs ($f^c = 80$)

Instances							% Gap with ADM	
ρ	$\nu_\bullet$	s	p_b	ADM	SBS	DLP	SBS	DLP
1.2	ν_H	5	0.4	68,294	51,455	66,540	24.656%	2.568%
		5	0.7	66,453	42,685	64,648	35.767%	2.716%
		25	0.4	67,583	51,455	66,103	23.864%	2.190%
		25	0.7	66,280	42,685	64,603	35.560%	2.530%
		50	0.4	66,541	51,455	65,239	22.671%	1.957%
		50	0.7	65,831	42,685	64,260	35.160%	2.386%
	ν_L	5	0.4	64,308	60,362	62,602	6.135%	2.653%
		5	0.7	63,595	58,056	61,922	8.710%	2.631%
		25	0.4	64,139	60,362	62,488	5.889%	2.574%
		25	0.7	63,514	58,056	61,846	8.594%	2.626%
		50	0.4	63,885	60,362	62,273	5.515%	2.523%
		50	0.7	63,401	58,056	61,738	8.431%	2.623%
1.6	ν_H	5	0.4	78,043	63,738	75,891	18.329%	2.757%
		5	0.7	76,541	56,267	74,315	26.488%	2.908%
		25	0.4	76,819	63,738	74,910	17.028%	2.485%
		25	0.7	76,067	56,267	74,069	26.030%	2.627%
		50	0.4	75,264	63,738	73,286	15.314%	2.628%
		50	0.7	75,385	56,267	73,413	25.361%	2.616%
	ν_L	5	0.4	74,220	71,411	72,162	3.786%	2.773%
		5	0.7	73,606	69,289	71,597	5.865%	2.729%
		25	0.4	73,943	71,411	72,013	3.424%	2.610%
		25	0.7	73,497	69,289	71,600	5.725%	2.581%
		50	0.4	73,540	71,411	71,633	2.895%	2.593%
		50	0.7	73,303	69,289	71,467	5.476%	2.505%

expensive fare class customers arrive later than the low-fare customers and the retain probability is high, FCM rejects the early commitment requests to keep seats for expensive fare class customers. Hence, it loses the potential revenue obtained from commitment reservations. Moreover, when p_b is low, we observe that allowing commitment arrivals during the whole planning period is not advantageous for FCM. Since the purchase probability is low, customers accepted towards the end of the booking horizon may result in empty seats.

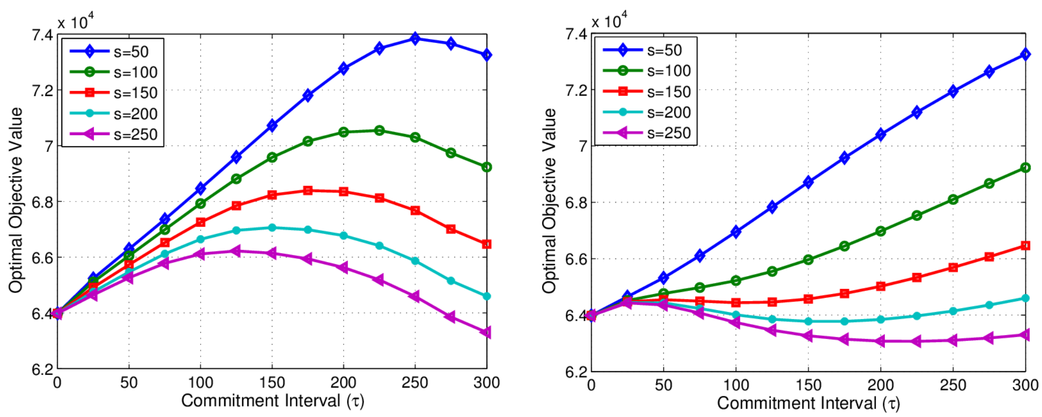
Next, we investigate the effect of the commitment period on the total expected revenue. Figure 5 plots the changes in optimal objective values of FCM and LCM with respect to different lengths of the commitment interval and the commitment period when p_b is low. Let τ^* denote the commitment interval value at which the maximum total expected revenue is obtained either by FCM or LCM in Figure 5. As the length of the commitment period (s) increases, the value of τ^* for FCM shifts to the beginning of the reservation period. Recall that the retain probability is positively correlated with the length of the



(a) $p_b = 0.25$ (b) $p_b = 0.75$

Figure 4: The effect of commitments on the total expected revenue for various buy probabilities

commitment period. Therefore, as s increases, the probability of waiting until the end of the reservation horizon (retain probability of a contingent commitment) also increases. As a result when p_b is low, it becomes more profitable for FCM to accept the commitment requests early in the reservation period and reserve seats for the late arrivals of expensive fare class customers. Similarly when s is high, it is more profitable for LCM to limit the commitment arrivals and τ^* for LCM shifts towards the end of the reservation period. However, even in this case, offering contingent commitment options can provide additional revenue compared to only offering standard booking products (when the commitment interval is 0).



(a) Total expected revenue of FCM (b) Total expected revenue of LCM

Figure 5: Change in the total expected revenue with respect to different s values (p_b=0.25)

In summary, Figure 4 depicts that accepting the commitment requests up to a certain time period is more profitable than accepting them during the whole reservation period when the purchase probability p_b of the contingent commitment option is low. As the purchase probability increases, allowing commitment arrivals during the whole period becomes more advantageous. In addition, offering a commitment option towards the end of the reservation period is more beneficial than offering it at the beginning of the reservation period when the purchase probability is high. Moreover, Figure 5 shows the effect of the length of the commitment period on the total revenue. We observe that as the length of the commitment period increases, it becomes more profitable to decrease the length of the time period during which the commitment option is offered.

6.3 An Alternate Simulation with Flexible Commitment Options. In this simulation study, we relax the assumption related to the purchase decision of the committed seats. In our proposed models, we assumed that the customers who committed to a seat make the buy or leave decision at the end of the commitment period. Although this is quite often the case, sometimes those customers may purchase the seat or leave at any time within the commitment period. In this section, we simulate such a setting. We implement ADM as in Section 4 without any changes. In our simulation, we assume that after committing to a seat, it is equally likely for a customer to make a decision in each one of the s periods. Since we compare the benchmark strategies for long commitment periods, we were not able to solve the DM model in this analysis. Our results are summarized in Table 4. The commitment fee is set to $f^c = 80$. Comparing the total expected revenues in Table 3 against those given in Table 4, we notice that the total net revenues obtained by the policies of all solution methods slightly improve in this alternate simulation. Since customers more frequently exercise options in the alternate simulation, the expected revenues obtained from the commitments increase. It is important to note that the percentage gaps between ADM and the other solution methods tend to increase when we allow customers to exercise their options at any time period. ADM adjusts the booking limits by taking into account the reservations and not exercised options that have already taken place. Therefore, it ends up accepting more reservation requests from lower fare classes than the deterministic model, and consequently, the revenue loss due to empty seats is counteracted by the gains from the committed seats.

We also analyze how our approximation performs. Figure 6 presents the gap between the optimal objective value of the approximate dynamic model and the average revenue obtained by its policy when regular and alternate simulations are run. The first observation we have is that the percentage gaps are small when the length of commitment period is short. The intuition behind this result is that, as the length increases, ADM fails to predict the dynamics of the commitment process. As a result, the number of empty seats increases and hence, its performance deteriorates. We caution the reader to the performances under the two simulation approaches. As Figure 6(a) depicts total revenue obtained in the alternate simulation is always higher than the one obtained in the regular simulation. Moreover, as the length of the commitment period increases, the performance of ADM worsens more than we expected. This result was more striking with our regular simulation. This behavior can be attributed to the structure of the alternate simulation. Since our approximation allows contingent commitments to exercise their options at any time, it performs better in the alternate simulation.

We conclude the presentation of our numerical results by reporting the CPU times of the proposed solution methods. We used a computer with 2.13 GHz Intel Pentium P6200 processor and 2 GB of RAM. The codes are written in MATLAB R2012b running under Windows 7 operating system. The intractable dynamic model requires on average 230 seconds for the problems where the length of commitment period is 5. It takes on average 120 seconds to solve the approximate dynamic model. DLP requires on average less than a second.

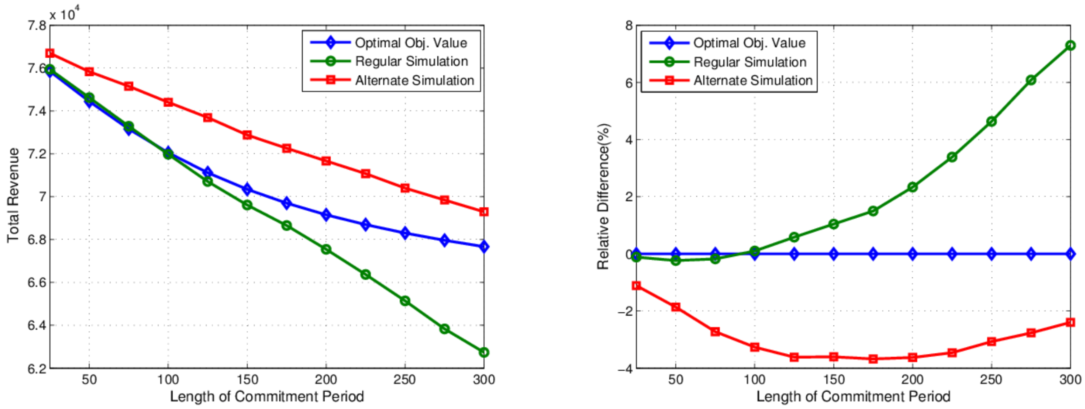
7. Conclusions. In this study, we introduce the concept of a commitment option. Recently such options have been offered by airline companies. By offering this option, they aim to attract price sensitive

Table 4: Computational results for the test problems in the alternate simulation ($f^c = 80$)

Instances							% Gap with ADM	
ρ	$\nu_\bullet$	s	p_b	ADM	SBS	DLP	SBS	DLP
1.2	ν_H	5	0.4	68,368	51,455	66,554	24.737%	2.653%
		5	0.7	66,488	42,685	64,667	35.800%	2.739%
		25	0.4	67,939	51,455	66,369	24.262%	2.311%
		25	0.7	66,465	42,685	64,693	35.779%	2.666%
		50	0.4	67,351	51,455	65,873	23.602%	2.194%
		50	0.7	66,220	42,685	64,540	35.541%	2.537%
	ν_L	5	0.4	64,333	60,362	62,607	6.172%	2.683%
		5	0.7	63,598	58,056	61,924	8.714%	2.632%
		25	0.4	64,249	60,362	62,574	6.050%	2.607%
		25	0.7	63,560	58,056	61,883	8.661%	2.638%
		50	0.4	64,114	60,362	62,476	5.852%	2.554%
		50	0.7	63,504	58,056	61,842	8.579%	2.617%
1.6	ν_H	5	0.4	78,151	63,738	75,931	18.442%	2.840%
		5	0.7	76,595	56,267	74,328	26.540%	2.960%
		25	0.4	77,499	63,738	75,338	17.756%	2.788%
		25	0.7	76,379	56,267	74,228	26.332%	2.816%
		50	0.4	76,650	63,738	74,360	16.844%	2.988%
		50	0.7	76,014	56,267	73,880	25.979%	2.807%
	ν_L	5	0.4	74,246	71,411	72,168	3.819%	2.800%
		5	0.7	73,615	69,289	71,599	5.877%	2.739%
		25	0.4	74,129	71,411	72,120	3.667%	2.710%
		25	0.7	73,590	69,289	71,648	5.844%	2.638%
		50	0.4	73,933	71,411	71,933	2.411%	2.705%
		50	0.7	73,496	69,289	71,627	5.724%	2.543%

customers as well as customers who have uncertainty in their travel time. We analyze the consequences of selling this option along with standard bookings of the products. We derive dynamic and static models for the capacity allocation problem. In the dynamic case, finding the optimal policy for the actual problem would require solving a dynamic program with a high-dimensional state vector. Thus, we propose an approximate dynamic programming formulation. In the deterministic case, we present a linear programming model leading to an upper bound on the optimal objective value of the actual problem.

We conduct a computational study to evaluate how offering options affects the airline's revenue. To assess the effect of commitment decisions, we compare the performances of our model against different policies. Our numerical results confirm the intuitive expectation that offering a commitment option is most beneficial when the purchase probability is high and the length of the commitment period is short. Furthermore, considering a policy that ignores the contingent option altogether, explicitly modeling the



(a) Total expected revenue (b) Percentage gap

Figure 6: The results related to optimal objective value of ADM and the average revenue obtained by the optimal policy of ADM

commitment option can bring significant revenue improvements even when the customer arrival intensity for this option is low. Also, making the contingent commitment option available up to only a certain time period can be more profitable than making it available during the whole sales horizon, especially when the purchase probability of the contingent commitment option is low. As the length of the commitment period increases, limiting the availability of the commitment option may be more beneficial to reap the most benefit from this option. Moreover, in our computational study we also evaluate the performance of our approximate dynamic model. When we compare our proposed model with the actual dynamic model, we see that there is no significant difference between their performances for the short commitment period length. This shows that our approximation performs very well.

An immediate extension of this work would be to include the overbooking option. Since commitment customers may not purchase their options and leave the system, the revenue loss due to the resulting empty seats can be filled by overbooking the flight. In this case, the overbooking limit should be determined by considering not exercising the option possibility of contingent commitments. Another future research direction is the application of the contingent commitment option to network problems. Network problems are quite difficult to solve due to the intractable state space. To overcome this difficulty, approximation methods based on decomposition are frequently proposed. However, in this case adverse effects of offering a contingent commitment option on the shared legs of the flight should be carefully investigated.

Appendix A. Omitted Proofs. We reserve this appendix for the proofs that we omitted in the main document.

Proposition 4.1 *Suppose the probability of a request being a commitment is class independent; that is, $\nu_1 = \nu_2 = \cdots = \nu_m$. Then, given the fare ordering $f_1 \geq f_2 \geq ... \geq f_m$, and hence, the ordering of the expected commitment revenues, $\phi_1 \geq \phi_2 \geq ... \geq \phi_m$, we have $u^*_{1t} \geq u^*_{2t} \geq ... \geq u^*_{mt}, t = 1, ..., T$.*

Proof. For given x and y, to maximize $V_t(x_t, y_t)$ we accept a booking or commitment request ($u^*_{it} = 1$) if

$$\begin{aligned} p_{it}(f_i + &\mathbb{E}V_{t+1}(x_t + 1 - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t))) + q_{it}(\phi_i + \mathbb{E}V_{t+1}(x_t + 1 - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t) + 1)) \\ &\geq p_{it}\mathbb{E}V_{t+1}(x_t - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t)) + q_{it}\mathbb{E}V_{t+1}(x_t - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t)). \end{aligned}$$

Let $\nu := \nu_1 = \nu_2 = \cdots = \nu_m$. Then, by using $p_{it} = (1-\nu)\alpha_{it}$ and $q_{it} = \nu\alpha_{it}$, we obtain

$$\begin{aligned} (1-\nu)(f_i + &\mathbb{E}V_{t+1}(x_t + 1 - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t))) + \nu(\phi_i + \mathbb{E}V_{t+1}(x_t + 1 - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t) + 1)) \\ &\geq \mathbb{E}V_{t+1}(x_t - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t)). \end{aligned} \tag{17}$$

Since $f_{i-1} \geq f_i$ and $\phi_{i-1} \geq \phi_i$, if relation (17) holds for a fare class i request ($u^*_{it} = 1$), then it also holds for the fare class $i-1$ request $u^*_{(i-1)t} = 1$. Similarly, if relation (17) does not hold for the expensive fare class $i-1$, then it does not hold for the cheaper fare class i either. This means that, if $u^*_{(i-1)t} = 0$ then $u^*_{it} = 0$. Therefore, we obtain the desired result. □

Proposition 5.1 *The optimal objective value of the DLP model gives an upper bound on the dynamic programming model (1a)-(1b). That is, $J_1(0, \mathbf{0}) \leq Z^*_{DLP}$.*

Proof. Suppose the random variables W_{it}, $\forall i, t$ denote the number of reservations accepted over the planning horizon under the optimal policy of the dynamic programming model. Each accepted reservation for fare class i either buys the contingent commitment option with probability ν_i or books the seat with probability $(1-\nu_i)$. Let X_{it} and Z_{it} be the random numbers of bookings and commitments accepted for fare class i at time period t, respectively. Since an accepted commitment request can leave with probability p_l, we also let S_{it} and L_{it} be the binary random numbers denoting the sold (exercised) and not exercised commitments, respectively. That is, S_{it} takes value 1, if there is a commitment reservation for fare class i at time period t and this commitment customer decides to exercise the option, and L_{it} takes value 1 if this commitment reservation leaves. As a result, $X_{it} + Z_{it} = W_{it}$ and $S_{it} + L_{it} = Z_{it}$ for all i, t.

Let now D_{it} be the random number of reservation requests for fare class i at time period t. Then, we have,

$$\mathcal{V}_1 = C, \tag{18}$$

$$\mathcal{V}_t = \mathcal{V}_{t-1} - \sum_{i=1}^{m} X_{i(t-1)} - \sum_{i=1}^{m} Z_{i(t-1)}, \qquad 2 \leq t \leq s+1, \tag{19}$$

$$\mathcal{V}_t = \mathcal{V}_{t-1} - \sum_{i=1}^{m} X_{i(t-1)} - \sum_{i=1}^{m} Z_{i(t-1)} + \sum_{i=1}^{m} L_{i(t-s-1)}, \quad s+2 \leq t \leq T, \tag{20}$$

$$\mathcal{V}_{T+1} = \mathcal{V}_T - \sum_{i=1}^m X_{iT} - \sum_{i=1}^m Z_{iT} + \sum_{k=T-s}^T \sum_{i=1}^m L_{ik}, \tag{21}$$

$$X_{it} + Z_{it} \le D_{it}, \qquad i = 1,...,m; t = 1,...,T, \tag{22}$$

$$\sum_{i=1}^m X_{it} + \sum_{i=1}^m Z_{it} \le \mathcal{V}_t, \quad t = 1,...,T, \tag{23}$$

where (18)-(21) ensure that the balance equations in each time period holds, (22) ensures that total number of bookings and commitments that we accept under the optimal policy do not exceed the reservation requests. Similarly, (23) guarantees that the total number of bookings and commitments that we accept do not exceed the available capacity. Consequently, the total revenue under the optimal policy of the dynamic programming is

$$\sum_{t=1}^T \sum_{i=1}^m f_i X_{it} + \sum_{t=1}^T \sum_{i=1}^m f^c Z_{it} + \sum_{t=1}^T \sum_{i=1}^m f_i S_{it}.$$

By conditioning on W_{it} we trivially obtain $\mathbb{E}(Z_{it}) = \nu\mathbb{E}(W_{it})$. Since $X_{it} = W_{it} - Z_{it}$, we have $\mathbb{E}(X_{it}) = (1-\nu_i)\mathbb{E}(W_{it})$. Similarly, conditioning on Z_{it} leads to $\mathbb{E}(S_{it}) = p_b\nu_i\mathbb{E}(W_{it})$. Therefore, the total expected revenue is given by

$$J_1(0,\mathbf{0}) = \sum_{t=1}^T \sum_{i=1}^m f_i(1-\nu_i)\mathbb{E}(W_{it}) + \sum_{t=1}^T \sum_{i=1}^m f^c \nu_i \mathbb{E}(W_{it}) + \sum_{t=1}^T \sum_{i=1}^m f_i p_b \nu_i \mathbb{E}(W_{it}).$$

Taking the expectations (18)-(23) and noting that $\mathbb{E}(D_{it}) = \alpha_{it}$, the solution given by $w_{it} = \mathbb{E}(W_{it})$ and $\vartheta_t = \mathbb{E}(\mathcal{V}_t)$ is feasible for the DLP model (4)-(12). Therefore, we have

$$Z^*_{DLP} \ge J_1(0,\mathbf{0}) = \sum_{t=1}^T \sum_{i=1}^m f_i(1-\nu_i)\mathbb{E}(W_{it}) + \sum_{t=1}^T \sum_{i=1}^m \phi_i \nu_i \mathbb{E}(W_{it}),$$

and the desired result holds. □

To prove the asymptotic bound result in Proposition 5.2, we first define a lower bound on the rate of convergence. Let d_{ib} and d_{ic} denote the random numbers of total fare class i requests for bookings and commitments, respectively. Then, the expected demands are computed as $\mu_i^b := \mathbb{E}(d_{ib}) = (1-\nu_i)\sum_{t=1}^T \alpha_{it}$ and $\mu_i^c := \mathbb{E}(d_{ic}) = \nu_i \sum_{t=1}^T \alpha_{it}$. Likewise, σ_{ib} and σ_{ic} denote the corresponding standard deviations. Then, the coefficient of variation of the number of requests for bookings and commitments are given as

$$CV_i^b = \frac{\sqrt{\sigma_{ib}^2}}{\mu_i^b} \text{ and } CV_i^c = \frac{\sqrt{\sigma_{ic}^2}}{\mu_i^c}, \text{ for } i = 1,...,m.$$

We also define

$$CV = \max_{1\le i\le m} \left\{CV_i^b, CV_i^c\right\},$$

as the maximum coefficient of variation.

Proposition A.1 *Let CV denote the maximum coefficient of variation over bookings and commitments for all fare classes. Then for $\epsilon \in [1-p_b, 1]$, we have $J_1(0,\mathbf{0}) \ge \left(1-\epsilon-\frac{CV^2}{\epsilon^2}\right)Z^*_{DLP-UB}$.*

Proof. Let $\{w^*_{it} : \forall i, t\}$ be the optimal value of the decision variables in problem (13)-(16). We consider a policy π that accepts at most $(1-\epsilon)(1-\nu_i)\sum_{t=1}^T w^*_{it}$ booking requests and $(1-\epsilon)\nu_i \sum_{t=1}^T w^*_{it}$ contingent commitment requests for fare class i for $\epsilon \in (0,1)$. Due to the capacity constraint (14) in DLP-UB model, the policy π is feasible if $(1-\epsilon) \le p_b$. The expected revenue $\mathcal{P}^\pi$ is given by

$$\mathcal{P}^\pi = \mathbb{E}[\sum_{i=1}^m f_i \min(d_{ib}, (1-\epsilon)\sum_{t=1}^T (1-\nu_i)w^*_{it}) + \sum_{i=1}^m f^c \min(d_{ic}, (1-\epsilon)\sum_{t=1}^T \nu_i w^*_{it}) +$$

$$\sum_{i=1}^{m} f_i \mathcal{S}(\min(d_{ic}, (1-\epsilon)\sum_{t=1}^{T} \nu_i w_{it}^*))],$$

where $\mathcal{S}(k)$ is a binomial random variable with k independent trials with success probability p_b and it gives the number of purchased committed seats. A lower bound to the generic term in the expression for $\mathcal{P}^\pi$ is then given by

$$\mathbb{E}[\min(d_{ic}, (1-\epsilon)\sum_{t=1}^{T} \nu_i w_{it}^*)] \geq (1-\epsilon)\sum_{t=1}^{T} \nu_i w_{it}^* \mathbf{P}(d_{ic} \geq (1-\epsilon)\sum_{t=1}^{T} \nu_i w_{it}^*) \tag{24}$$

$$\geq (1-\epsilon)\sum_{t=1}^{T} \nu_i w_{it}^* \mathbf{P}(d_{ic} \geq (1-\epsilon)\sum_{t=1}^{T} \nu_i \alpha_{it}) \tag{25}$$

$$\geq (1-\epsilon)\sum_{t=1}^{T} \nu_i w_{it}^* \Big(1 - \frac{CV_i^{c2}}{CV_i^{c2} + \epsilon^2}\Big) \tag{26}$$

$$\geq (1-\epsilon)\sum_{t=1}^{T} \nu_i w_{it}^* \Big(1 - \frac{CV^2}{\epsilon^2}\Big). \tag{27}$$

The inequality (25) holds since $\nu_i w_{it}^* \leq \nu_i \alpha_{it}$, (26) follows from the Marshall's inequality and (27) holds due to the definition of CV. Since $\mathbb{E}[\mathcal{S}(\min(d_{ic}, (1-\epsilon)\sum_{t=1}^{T} \nu_i w_{it}^*))] = p_b \mathbb{E}[\min(d_{ic}, (1-\epsilon)\sum_{t=1}^{T} \nu_i w_{it}^*)]$, we can give a lower bound to $\mathcal{P}^\pi$ by using the inequality (27) as follows:

$$\begin{aligned}
\mathcal{P}^\pi &\geq (1-\epsilon)\Big(1 - \frac{CV^2}{\epsilon^2}\Big)(\sum_{i=1}^{m}\sum_{t=1}^{T} f_i(1-\nu_i)w_{it}^* + \sum_{i=1}^{m}\sum_{t=1}^{T} f^c \nu_i w_{it}^* + \sum_{i=1}^{m}\sum_{t=1}^{T} f_i p_b \nu_i w_{it}^*) \\
&\geq (1-\epsilon)\Big(1 - \frac{CV^2}{\epsilon^2}\Big) Z_{DLP-UB}^* \\
&\geq \Big(1 - \epsilon - \frac{CV^2}{\epsilon^2}\Big) Z_{DLP-UB}^*
\end{aligned}$$

This implies

$$J_1(0, \mathbf{0}) \geq \mathcal{P}^\pi \geq \Big(1 - \epsilon - \frac{CV^2}{\epsilon^2}\Big) Z_{DLP-UB}^*.$$

□

To tighten the lower bound in the above inequality, we maximize it over ϵ and obtain $\epsilon^* = \max\{(2CV^2)^{1/3}, 1-p_b\}$. Since $\epsilon \in [1-p_b, 1]$, this tighter bound is only obtained when $2CV^2 < 1$. Consequently we have,

$$J_1(0, \mathbf{0}) \geq \mathcal{P}^\pi \geq \Big(1 - \epsilon^* - \frac{CV^2}{\epsilon^{*2}}\Big) Z_{DLP-UB}^*.$$

Next we examine the structure of the lower bound as the problem size gets large.

PROPOSITION 5.2 *Given $\epsilon \in [1-p_b, 1]$ and $\kappa > 0$, we have*

$$Z_{DLP-UB}^\kappa \geq Z_{DLP}^\kappa \geq J_1^\kappa(0, \mathbf{0}) \geq \left(1 - \epsilon - \frac{CV^2}{\kappa\epsilon^2}\right) Z_{DLP-UB}^\kappa,$$

where CV denotes the maximum coefficient of variation over bookings and commitments for all fare classes. Therefore,

$$p_b \leq \lim_{\kappa\to\infty} \frac{J_1^\kappa(0, \mathbf{0})}{Z_{DLP}^\kappa} \leq 1.$$

PROOF. We observe that if $\{w_{it}^* : \forall i, t\}$ is an optimal solution to problem (13)-(16). Then $\{w_{i\lceil t/\kappa\rceil}^* : \forall i, t\}$ is an optimal solution for the scaled problem. Thus, it follows that $Z_{DLP-UB}^\kappa = \kappa Z_{DLP-UB}$. For the scaled problems, the expected demand and the variance are scaled with κ. If μ and σ^2 denote the mean demand and variance for problem $\mathcal{P}^1$, then the mean demand is $\kappa\mu$ and the variance is $\kappa\sigma^2$ for the problem $\mathcal{P}^\kappa$. Therefore the maximum coefficient of variation of the scaled problem is

$$CV^\kappa = \max_{1\leq i\leq m}\left\{\frac{\sqrt{\kappa\sigma_{ib}^2}}{\kappa\mu_i^b}, \frac{\sqrt{\kappa\sigma_{ic}^2}}{\kappa\mu_i^c}\right\} = \frac{CV\sqrt{\kappa}}{\kappa}$$

By following the result of Proposition A.1 and replacing CV^κ with $\frac{CV\sqrt{\kappa}}{\kappa}$, we have

$$J_1^\kappa(0,\mathbf{0}) \geq \Big(1-\epsilon-\frac{CV^2}{\kappa\epsilon^2}\Big)Z_{DLP-UB}^\kappa.$$

When κ goes to infinity, the expression $\Big(1-\epsilon-\frac{CV^2}{\kappa\epsilon^2}\Big)$ approaches to $(1-\epsilon)$. Since $\epsilon\in[1-p_b,1]$, this bound is tighter when $\epsilon = 1-p_b$. Therefore, as p_b goes to 1, the upper bound obtained from Z_{DLP-UB}^* becomes asymptotically tight. Following the result of Proposition A.1, we obtain the following convergence rate

$$\kappa Z_{DLP-UB} \geq Z_{DLP}^\kappa \geq J_1^\kappa(0,\mathbf{0}) \geq \Big(1-\epsilon-\frac{CV^2}{\kappa\epsilon^2}\Big)\kappa Z_{DLP-UB},$$

Dividing the chain of inequalities by κZ_{DLP-UB} and taking the limit as κ goes to infinity, we get

$$p_b \leq \lim_{\kappa\to\infty}\frac{J_1^\kappa(0,\mathbf{0})}{Z_{DLP-UB}^\kappa} \leq \lim_{\kappa\to\infty}\frac{Z_{DLP}^\kappa}{Z_{DLP-UB}^\kappa} \leq 1$$

which implies,

$$p_b \leq \lim_{\kappa\to\infty}\frac{J_1^\kappa(0,\mathbf{0})}{Z_{DLP}^\kappa} \leq 1.$$

□

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