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IDENTIFYING ACTIVITY*

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Abstract. Identification of active constraints in constrained optimization is of interest from both practical and theoretical viewpoints, as it holds the promise of reducing an inequality-constrained problem to an equality-constrained problem, in a neighborhood of a solution. We study this issue in the more general setting of composite nonsmooth minimization, in which the objective is a composition of a smooth vector function c with a lower semicontinuous function h, typically nonsmooth but structured. In this setting, the graph of the generalized gradient ∂h can often be decomposed into a (nondisjoint) union of simpler subsets. "Identification" amounts to deciding which subsets of the graph are "active" in the criticality conditions at a given solution. We give conditions under which any convergent sequence of approximate critical points finitely identifies the activity. Prominent among these properties is a condition akin to the Mangasarian–Fromovitz constraint qualification, which ensures boundedness of the set of multiplier vectors that satisfy the optimality conditions at the solution.

Key words. constrained optimization, composite optimization, Mangasarian–Fromovitz constraint qualification, active set, identification

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1. First-order conditions for composite optimization.

The composite optimization model

$$\min_x h(c(x)), \tag{1.1}$$

for extended-valued functions $h:\mathbb{R}^m \to \bar{\mathbb{R}}$ and smooth vector functions $c:\mathbb{R}^n \to \mathbb{R}^m$, unifies a wide variety of interesting concrete optimization problems. In applications, the function h typically has a simple structure, and it is often convex. Here we consider the general case, while noting throughout how convexity simplifies matters. Assuming the inner function c is everywhere defined simplifies notation; the case where its domain is an open subset of $\mathbb{R}^n$ is a trivial extension. Because h can take values in the extended reals $\bar{\mathbb{R}} = [-\infty, +\infty]$, we can easily model constraints.

In this work we study "active-set" ideas in this composite framework. In classical nonlinear programming, if an algorithm can identify which inequality constraints hold with equality at the optimal solution, then the problem reduces to the simpler case of equality-constrained optimization. The identification problem is surveyed extensively in [6], where the authors also describe an elegant general technique for solving it exactly without assuming strict complementarity or uniqueness of Lagrange multipliers, relying only on the Mangasarian–Fromovitz constraint qualification.

Our interest here is the more general composite optimization problem (1.1), rather than classical nonlinear programming. Like [6], we rely only on the Mangasarian–

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Fromovitz condition (or rather its appropriate generalization). However, unlike [6], our aim is not to solve the identification problem exactly; instead we study an easier problem, which in nonlinear programming amounts to identifying a "sufficient" set of the constraints to reduce to equality-constrained optimization. Our approach is an extensive generalization of the nonlinear programming case developed in [16]. Our main result is rather easy to prove: rather than refining the classical case, its intent is to provide a simple conceptual framework for more general active-set techniques.

We begin with a quick review of the first-order optimality conditions, which rely on standard ideas from nonsmooth optimization and variational analysis. We refer to the monographs [5], [20], [14], [15] for more details. In particular, we follow the notation and terminology of [20].

Consider a point y in a closed subset S of the space $\mathbb{R}^n$, endowed with its usual Euclidean norm. A vector $w \in \mathbb{R}^n$ is *tangent* to S at y if there exists a sequence of points $y^r \in S$ with $y^r \to y$ and a sequence of scalars $\tau_r \in \mathbb{R}_+$ such that $\tau_r(y^r - y) \to w$. The cone of such tangent vectors is denoted $T_S(y)$. On the other hand, z is *normal* to S at y if there exist sequences of points $y^r \to y$ and $u^r \to y$, and a sequence of scalars $\tau_r \in \mathbb{R}_+$, such that each u^r is a nearest point to y^r in S and $\tau_r(y^r - u^r) \to z$. The set of such normal vectors is denoted $N_S(y)$. (This definition, in fact, describes the "limiting proximal normal cone"; in our finite-dimensional setting it coincides with the normal cone of [5], [20], [14], [15].) When S is convex, the normal cone coincides with the classical normal cone of convex analysis, and when S is a smooth manifold, it coincides with the classical normal space. If $\langle w, z\rangle \le 0$ for all $w \in T_S(y)$ and $z \in N_S(y)$, as holds, in particular, if S is convex or a smooth manifold, then S is called *Clarke regular* at y.

For a lower semicontinuous function $h\colon \mathbb{R}^m \to \bar{\mathbb{R}}$ with epigraph

$$\operatorname{epi} h = \{(c, \rho) \in \mathbb{R}^m \times \mathbb{R} \colon \rho \ge h(c)\},$$

at any point $\bar{c} \in \mathbb{R}^m$ where the value $h(\bar{c})$ is finite, the *subdifferential* and *horizon subdifferential* are defined by

$$\partial h(\bar{c}) = \{w \in \mathbb{R}^m \colon (w, -1) \in N_{\operatorname{epi} h}(\bar{c}, h(\bar{c}))\}, \tag{1.2a}$$

$$\partial^\infty h(\bar{c}) = \{w \in \mathbb{R}^m \colon (w, 0) \in N_{\operatorname{epi} h}(\bar{c}, h(\bar{c}))\}, \tag{1.2b}$$

respectively. (If $h(\bar{c})$ is infinite, we define $\partial h(\bar{c}) = \varnothing$ and $\partial h^\infty(\bar{c}) = \{0\}$.) The point $\bar{c}$ is *critical* if $0 \in \partial h(\bar{c})$. If h is convex, $\partial h(\bar{c})$ coincides with the classical object of convex analysis, and $\partial^\infty h(\bar{c})$ is the normal cone to the domain of h at $\bar{c}$. If, on the other hand, h is smooth at $\bar{c}$, then $\partial h(\bar{c}) = \{\nabla h(\bar{c})\}$, and $\partial^\infty h(\bar{c}) = \{0\}$. The function h is called *subdifferentially regular* at $\bar{c}$ if $\operatorname{epi} h$ is Clarke regular at the point $(\bar{x}, h(\bar{c}))$, as holds, in particular, if h is convex or smooth.

Throughout this work, we make the following rather standard blanket assumption.

Assumption 1. The function $h\colon \mathbb{R}^m \to \bar{\mathbb{R}}$ is lower semicontinuous and the function $c\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable. The point $\bar{x} \in \mathbb{R}^m$ is critical for the composite function $h \circ c$, and it satisfies the condition

$$\partial^\infty h(c(\bar{x})) \cap N(\nabla c(\bar{x})^*) = \{0\}. \tag{1.3}$$

Here, the map $\nabla c(\bar{x})\colon \mathbb{R}^n \to \mathbb{R}^m$ is the Jacobian mapping, * denotes the adjoint, and $N(\cdot)$ denotes the null space. Equation (1.3) is called a regularity (or *transversality*) condition.

Assumption 1 implies, via a standard chain rule [20, Theorem 10.6], the existence of a vector $v \in \mathbb{R}^m$ satisfying the conditions

$$v \in \partial h(c(\bar{x})), \qquad \nabla c(\bar{x})^* v = 0. \tag{1.4}$$

By analogy with classical nonlinear programming (as we shall see), we call a vector $v \in \mathbb{R}^m$ satisfying the conditions (1.4) a *multiplier vector* for the critical point $\bar{x}$.

In seeking to solve the problem (1.1), we thus focus attention on pairs $(x, v) \in \mathbb{R}^n \times \mathbb{R}^m$ solving the system

$$(c(x), v) \in \operatorname{gph}(\partial h), \qquad \nabla c(x)^* v = 0, \tag{1.5}$$

written in terms of the graph of the subdifferential of h. Assumption 1 implies that this system is solvable with $x = \bar{x}$. Conversely, if the function h is subdifferentially regular at the point $c(x)$, then this system implies that x is a critical point of the composite function $h \circ c$, via another standard chain rule [20, Theorem 10.6].

Solving the system (1.5) is often difficult in part because the graph $\operatorname{gph}(\partial h)$ may have a complicated structure. Active-set methods from classical nonlinear programming and its extensions essentially restrict attention to a suitable subset of $\operatorname{gph}(\partial h)$, thereby narrowing a local algorithmic search for a critical point. We therefore make the following definition.

Definition 1.1. *An* actively sufficient set *for a critical point $\bar{x}$ of the composite function $h \circ c$ is a set $G \subset \operatorname{gph}(\partial h)$ containing a point of the form $(c(\bar{x}), \bar{v})$, where $\bar{v}$ is a multiplier vector for $\bar{x}$.*

The central idea we explore in this work is how to "identify" actively sufficient sets from among the parts of a decomposition of the graph $\operatorname{gph}(\partial h)$. We present conditions ensuring that any sufficiently accurate approximate solution of system (1.5) with the pair $[x, h(c(x))]$ sufficiently near the pair $[\bar{x}, h(c(\bar{x}))]$ identifies an actively sufficient set.

2. The case of classical nonlinear programming.

To illustrate the abstract composite optimization framework, we consider the special case of classical nonlinear programming, which we state as follows:

$$\text{(NLP)} \qquad \begin{cases} \inf & f(x) \\ \text{subject to} & p_i(x) = 0 \quad (i = 1, 2, \ldots, s), \\ & q_j(x) \le 0 \quad (j = 1, 2, \ldots, t), \\ & x \in \mathbb{R}^n, \end{cases}$$

where the functions $f, p_i, q_j : \mathbb{R}^n \to \mathbb{R}$ are all continuously differentiable.

We can model the problem (NLP) in our composite form (1.1) by defining a continuously differentiable function $c : \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^s \times \mathbb{R}^t$ and a polyhedral function $h : \mathbb{R} \times \mathbb{R}^s \times \mathbb{R}^t \to \bar{\mathbb{R}}$ through

$$c(x) = (f(x), p(x), q(x)) \qquad (x \in \mathbb{R}^n), \tag{2.1a}$$

$$h(u, y, w) = \begin{cases} u & (y = 0, w \le 0) \\ +\infty & (\text{otherwise}) \end{cases} \qquad (u \in \mathbb{R}, y \in \mathbb{R}^s, w \in \mathbb{R}^t). \tag{2.1b}$$

Clearly for any point $x \in \mathbb{R}^n$, the adjoint map $\nabla c(x)^* : \mathbb{R} \times \mathbb{R}^s \times \mathbb{R}^t \to \mathbb{R}^n$ is given by

$$\nabla c(x)^*(\theta,\lambda,\phi) = \theta\nabla f(x) + \sum_i \lambda_i \nabla p_i(x) + \sum_j \phi_j \nabla q_j(x).$$

The subdifferential and horizon subdifferential of h at any point $(u,0,w) \in \mathbb{R}\times\mathbb{R}^s\times\mathbb{R}^t_-$ are given by

$$\partial h(u,0,w) = \{1\}\times\mathbb{R}^s\times\{\phi\in\mathbb{R}^t_+ : \langle\phi,w\rangle = 0\},$$
$$\partial^\infty h(u,0,w) = \{0\}\times\mathbb{R}^s\times\{\phi\in\mathbb{R}^t_+ : \langle\phi,w\rangle = 0\}.$$

(Elsewhere in $\mathbb{R}\times\mathbb{R}^s\times\mathbb{R}^t$, these two sets are, respectively, $\varnothing$ and $\{0\}$.)

Armed with these calculations, consider any critical point $\bar{x}$ (or, in particular, any local minimizer for the nonlinear program). By assumption, $\bar{x}$ is a feasible solution. Classically, the *active set* is

$$\bar{J} = \{j : q_j(\bar{x}) = 0\}. \tag{2.2}$$

The regularity condition (1.3) becomes the following assumption.

Assumption 2 (Mangasarian–Fromovitz). The only pair $(\lambda,\phi)\in\mathbb{R}^s\times\mathbb{R}^t_+$ satisfying $\phi_j = 0$ for $j\notin\bar{J}$ and

$$\sum_i \lambda_i\nabla p_i(\bar{x}) + \sum_j \phi_j \nabla q_j(\bar{x}) = 0$$

is $(\lambda,\phi) = (0,0)$.

In this framework, what we have called a multiplier vector for the critical point $\bar{x}$ is just a triple $(1,\bar{\lambda},\bar{\phi})\in\mathbb{R}\times\mathbb{R}^s\times\mathbb{R}^t_+$ satisfying $\bar{\phi}_j = 0$ for $j\notin\bar{J}$ and

$$\nabla f(\bar{x}) + \sum_i \bar{\lambda}_i\nabla p_i(\bar{x}) + \sum_j \bar{\phi}_j\nabla q_j(\bar{x}) = 0. \tag{2.3}$$

It is evident that the solvability of the system (1.4) retrieves the classical first-order criticality conditions: existence of Lagrange multipliers under the Mangasarian–Fromovitz constraint qualification.

Nonlinear programming is substantially more difficult than solving nonlinear systems of equations, because we do not know the active set $\bar{J}$ in advance. Active-set methods try to identify $\bar{J}$, since, once this set is known, we can find a stationary point by solving the system

$$\begin{aligned} \nabla f(x) + \sum_i \lambda_i\nabla p_i(x) + \sum_{j\in\bar{J}} \phi_j\nabla q_j(x) &= 0, \\ p_i(x) &= 0 \quad (i = 1,2,\ldots,p), \\ q_j(x) &= 0 \quad (j\in\bar{J}), \end{aligned}$$

which is a nonlinear system of $n+p+|\bar{J}|$ equations for the vector $(x,\lambda,\phi_{\bar{J}})\in\mathbb{R}^n\times\mathbb{R}^p\times\mathbb{R}^{|\bar{J}|}$. Our aim here is to formalize this process of identification, generalizing the approach of [16] to the broader framework of composite minimization.

3. Main result. We start with a useful tool.

LEMMA 3.1. *Under Assumption* 1*, the set of multiplier vectors for $\bar{x}$ is nonempty and compact.*

Proof. We have already observed the existence of a multiplier vector. Since the subdifferential $\partial h(c(\bar{x}))$ is a closed set, the set of multipliers must also be closed. Assuming for contradiction that this set is unbounded, we can find a sequence $\{v^r\}$ with $|v^r| \to \infty$ and

$$v^r \in \partial h(c(\bar{x})), \qquad \nabla c(\bar{x})^* v^r = 0.$$

By defining $w^r := v^r/|v^r|$, we have $|w^r| \equiv 1$, and hence without loss of generality we can assume $w^r \to \bar{w}$ with $|\bar{w}| = 1$. Clearly, since $w^r \in N(\nabla c(\bar{x})^*)$ and the null space is closed, we have $\bar{w} \in N(\nabla c(\bar{x})^*)$. On the other hand, $\bar{w} \in \partial^\infty h(c(\bar{x}))$ follows from the definition of the horizon subdifferential. Since $\bar{w} \neq 0$ we have a contradiction to condition (1.3). ☐

We are ready to present the main result.

THEOREM 3.2. *Suppose Assumption* 1 *holds, and consider any closed set* $G \subset \mathrm{gph}(\partial h)$. *There exists a number* $\epsilon > 0$ *such that if a sequence of points* $x^r \to \bar{x}$ *and pairs* $(c^r, v^r) \in \mathrm{gph}(\partial h)$ *satisfy* $c^r \to c(\bar{x})$, $h(c^r) \to h(c(\bar{x}))$, *and* $\nabla c(x^r)^* v^r \to 0$, *and if furthermore* $\mathrm{dist}((c^r, v^r), G) < \epsilon$ *for infinitely many* $r = 1, 2, 3, \ldots$, *then* G *is an actively sufficient set for* $\bar{x}$.

Proof. Suppose the result fails. Then G is not an actively sufficient set, and yet there exists a sequence of strictly positive numbers $\epsilon_j \downarrow 0$ as $j \to \infty$ such that, for each $j = 1, 2, \ldots$, the following property holds. There exist sequences

$$x_j^r \in \mathbb{R}^n, \quad c_j^r \in \mathbb{R}^m, \quad v_j^r \in \partial h(c_j^r), \quad r = 1, 2, \ldots,$$

satisfying

$$x_j^r \to \bar{x}, \quad c_j^r \to c(\bar{x}), \quad h(c_j^r) \to h(c(\bar{x})), \quad \nabla c(x_j^r)^* v_j^r \to 0,$$

as $r \to \infty$, and yet

$$\mathrm{dist}((c_j^r, v_j^r), G) < \epsilon_j, \qquad r = 1, 2, \ldots.$$

To prove boundedness of $\{v_j^r\}_{r=1}^\infty$ for each j, we use a slight extension of the proof of Lemma 3.1. Supposing for contradiction that $|v_j^r| \to_r \infty$, we define $w^r := v_j^r/|v_j^r|$; thus $|w^r| \equiv 1$, and without loss of generality $w^r \to \bar{w}$ with $|\bar{w}| = 1$. We have from $\nabla c(x_j^r)^* w^r \to_r 0$ and from continuity of ∇c that $\bar{w} \in N(\nabla c(\bar{x})^*)$. Further, by (1.2), and outer semicontinuity of $N_{\mathrm{epi}\, h}$ (see [20, Proposition 6.6]), we have

$$\begin{aligned} v_j^r \in \partial h(c_j^r) &\Rightarrow (v_j^r, -1) \in N_{\mathrm{epi}\, h}(c_j^r, h(c_j^r)) \Rightarrow (w^r, -1/|v_j^r|) \in N_{\mathrm{epi}\, h}(c_j^r, h(c_j^r)) \\ &\Rightarrow (\bar{w}, 0) \in N_{\mathrm{epi}\, h}(c(\bar{x}), h(c(\bar{x}))) \Rightarrow \bar{w} \in \partial^\infty h(c(\bar{x})), \end{aligned}$$

which contradicts (1.3). For each j, we can therefore assume without loss of generality that the sequence $\{v_j^r\}_{r=1}^\infty$ converges to some vector v_j, which must be a multiplier vector at $\bar{x}$. By continuity, we deduce

$$\mathrm{dist}((c(\bar{x}), v_j), G) \le \epsilon_j.$$

By an argument similar to the one above, the sequence $\{v_j\}_{j=1}^\infty$ is bounded, so after taking a subsequence of the indices j, we can suppose that it converges to some multiplier vector $\bar{v}$. Noting that the set G is closed, we have by taking limits as $j \to \infty$ that $(c(\bar{x}), \bar{v}) \in G$, contradicting the assumption that G is not an actively sufficient set. ☐

An easy corollary extends from one potential actively sufficient set to many.

COROLLARY 3.3. *Suppose Assumption* 1 *holds, and consider any finite family* $\mathcal{G}$ *of closed subsets of* $\mathrm{gph}(\partial h)$*. There exists a number* $\epsilon > 0$ *such that if a sequence of points* $x^r \to \bar{x}$ *and pairs* $(c^r, v^r) \in \mathrm{gph}(\partial h)$ *satisfy* $c^r \to c(\bar{x})$*,* $h(c^r) \to h(c(\bar{x}))$*, and* $\nabla c(x^r)^* v^r \to 0$*, then any set* $G \in \mathcal{G}$ *satisfying* $\mathrm{dist}((c^r, v^r), G) < \epsilon$ *for infinitely many* $r = 1, 2, 3, \ldots$ *is an actively sufficient set for* $\bar{x}$*.*

Proof. For each set $G \in \mathcal{G}$, we apply Theorem 3.2, deducing the existence of a number $\epsilon_G > 0$ such that the conclusion of the theorem holds. The desired result then holds for $\epsilon = \min_G \epsilon_G$. ∎

The following result is a simple special case, easily proved directly.

COROLLARY 3.4. *Under the assumptions of Corollary* 3.3*, there exists a number* $\tilde{\epsilon} > 0$ *such that*

$$\mathrm{dist}((c(\bar{x}), \bar{v}), G) > \tilde{\epsilon} \tag{3.1}$$

for all multiplier vectors $\bar{v}$ *for the critical point* $\bar{x}$*, and all sets* $G \in \mathcal{G}$ *that are not actively sufficient for* $\bar{x}$*.*

Proof. Recall that all $G \in \mathcal{G}$ are closed, so if G is not actively sufficient, we have $\mathrm{dist}((c(\bar{x}), \bar{v}), G) > \epsilon_G$ for some $\epsilon_G > 0$. We obtain the result by setting $\tilde{\epsilon}$ to the minimum of ϵ_G over the finitely many non-actively-sufficient sets $G \in \mathcal{G}$. ∎

We end this section with another corollary.

COROLLARY 3.5. *Suppose Assumption* 1 *holds. Consider any finite family* $\mathcal{G}$ *of closed subsets of* $\mathrm{gph}(\partial h)$*. Then for any sequence of points* $x^r \in \mathbb{R}^n$*, vectors* $c^r \in \mathbb{R}^m$*, subgradients* $v^r \in \partial h(c^r)$*, and sets* $G^r \in \mathcal{G}$ *(for* $r = 1, 2, \ldots$*), satisfying*

$$x^r \to \bar{x}, \quad |c^r - c(x^r)| \to 0, \quad h(c^r) \to h(c(\bar{x})),$$
$$\nabla c(x^r)^* v^r \to 0, \quad \mathrm{dist}((c^r, v^r), G^r) \to 0$$

as $r \to \infty$*, the set* G^r *is actively sufficient for* $\bar{x}$ *for all* r *sufficiently large.*

Proof. Apply Corollary 3.3. ∎

This last result is a little weaker than Corollary 3.3, which requires only that the distance $\mathrm{dist}((c^r, v^r), G^r)$ be sufficiently small, rather than shrinking to zero.

We note that the transversality condition (1.3) is crucial to the analysis of this section, its role being to ensure boundedness of the sequence $\{v^r\}$ of multipliers. It is interesting to consider whether identifiability results can be proved in the presence of a weaker condition, thereby making them applicable to such problems as optimization with equilibrium constraints, which usually have unbounded optimal multipler sets. In the context of nonlinear programming, some algorithms include devices to prevent the multiplier estimates from growing too large (see, for example, [22]). Extension of these approaches and their associated identification results to the more general setting of this paper is far from obvious, but worth exploring in future work.

4. Subdifferential graph decomposition.

To apply the ideas in the previous section, we typically assume the availability of a decomposition of $\mathrm{gph}(\partial h)$ (the graph of the subdifferential of h) into some finite union of closed, not necessarily disjoint sets $G^1, G^2, \ldots, G^k \subset \mathbb{R}^m \times \mathbb{R}^m$. For this decomposition to be useful, the sets G^i should be rather simple so that the restricted system

$$(c(x), v) \in G^i, \qquad \nabla c(x)^* v = 0$$

is substantially easier to solve than the original criticality system. The more refined the decomposition, the more information we may be able to derive from the identification process. Often we have in mind the situation where each of the sets G^i is a polyhedron. We might, for example, assume that whenever some polyhedron is contained in the list (G^i), so is its entire associated lattice of closed faces.

Example 4.1 (scalar examples). We give some simple examples in the case $m = 1$. Consider first the indicator function for $\mathbb{R}_+$, defined by $h(c) = 0$ for $c \geq 0$ and $+\infty$ for $c < 0$. We have

$$\partial h(c) = \begin{cases} \emptyset & \text{if } c < 0, \\ (-\infty, 0] & \text{if } c = 0, \\ \{0\} & \text{if } c > 0. \end{cases}$$

Thus an appropriate decomposition is $\mathrm{gph}(\partial h) = G^1 \cup G^2 \cup G^3$, where

$$G^1 = \{0\} \times (-\infty, 0], \quad G^2 = \{(0,0)\}, \quad G^3 = [0, \infty) \times \{0\}.$$

Similar examples are the absolute value function $|\cdot|$, for which a decomposition is $\mathrm{gph}(\partial|\cdot|) = G^1 \cup G^2 \cup G^3$, where

$$G^1 = (-\infty, 0] \times \{-1\}, \quad G^2 = \{0\} \times [-1, 1], \quad G^3 = [0, \infty) \times \{1\} \tag{4.1}$$

(further refinable by including the two sets $\{0, \pm 1\}$), and the positive-part function $(c) = \max(c, 0)$, for which a decomposition is $\mathrm{gph}(\partial \mathrm{pos}) = G^4 \cup G^5 \cup G^6$, where

$$G^4 = (-\infty, 0] \times \{0\}, \quad G^5 = \{0\} \times [0, 1], \quad G^6 = [0, \infty) \times \{1\} \tag{4.2}$$

(again refinable). A last scalar example, which involves a nonconvex function h, is given by $h(c) = 1 - e^{-\alpha|c|}$ for some constant $\alpha > 0$. We have

$$\partial h(c) = \begin{cases} \{-\alpha\, e^{\alpha c}\} & \text{if } c < 0, \\ [-\alpha, \alpha] & \text{if } c = 0, \\ \{\alpha\, e^{-\alpha c}\} & \text{if } c > 0. \end{cases}$$

An appropriate partition is $\mathrm{gph}(\partial h) = G^1 \cup G^2 \cup G^3$, where

$$G^1 = \{(c, -\alpha\, e^{\alpha c}) : c \leq 0\}, \quad G^2 = \{0\} \times [-\alpha, \alpha], \quad G^3 = \{(c, \alpha\, e^{-\alpha c}) : c \geq 0\}.$$

Example 4.2 (an $\ell 1$-penalty function). Consider a function $h : \mathbb{R}^2 \to \mathbb{R}$ that is an ℓ_1-penalty function for the constraint system $c_1 = 0$, $c_2 \leq 0$, that is,

$$h(c) = |c_1| + \max(c_2, 0). \tag{4.3}$$

Using the notation of the previous example, we have

$$\partial h(c_1, c_2) = \partial(|\cdot|)(c_1) \times \partial \mathrm{pos}(c_2).$$

A partition of $\mathrm{gph}(\partial h)$ into nine closed sets can be constructed by using interleaved Cartesian products of (4.1) and (4.2).

Example 4.3 (classical nonlinear programming). Consider once again the setup of section 2. The classical notion of active set in nonlinear programming arises from a certain combinatorial structure in the graph of the subdifferential ∂h of the outer function h

defined in (2.1b):

$$\operatorname{gph}(\partial h) = \{((u,0,w),(1,\lambda,\boldsymbol{\phi})) \colon w \le 0, \boldsymbol{\phi} \ge 0, \langle w, \boldsymbol{\phi}\rangle = 0\}. \tag{4.4}$$

We can decompose this set into a finite union of polyhedra as follows:

$$\operatorname{gph}(\partial h) = \bigcup_{J \subset \{1,2,\ldots,t\}} G^J,$$

where

$$G^J = \{((u,0,w),(1,\lambda,\boldsymbol{\phi})) \colon w \le 0, \boldsymbol{\phi} \ge 0, w_j = 0\,(j \in J), \boldsymbol{\phi}_j = 0\,(j \notin J)\}. \tag{4.5}$$

According to our definition, G^J is an actively sufficient set exactly when $J \subset \bar{J}$ (defined in (2.2)), and there exist vectors $\bar{\lambda} \in \mathbb{R}^s$ and $\bar{\boldsymbol{\phi}} \in \mathbb{R}^t_+$ satisfying the stationarity condition (2.3) with $\bar{\boldsymbol{\phi}}_j = 0$ for all $j \notin J$. We call such an index set J *sufficient* at $\bar{x}$. We pursue this example further in the next section.

Much interest lies in the case in which the function h is polyhedral so that $\operatorname{gph}(\partial h)$ is a finite union of polyhedra. However, the latter property holds more generally for the "piecewise linear-quadratic" functions defined in [20]; see, in particular, [20, Proposition 10.21].

Of course, we cannot decompose the graph of the subdifferential ∂h into a finite union of closed sets unless this graph is itself closed. This property may fail, even for quite simple functions. For example, the lower semicontinuous function $h \colon \mathbb{R} \to \mathbb{R}$ defined by $h(c) = 0$ for $c \le 0$ and $h(c) = 1 - c$ for $c > 0$ has subdifferential given by

$$\partial h(c) = \begin{cases} \{0\} & \text{if } c < 0, \\ [0,\infty) & \text{if } c = 0, \\ \{-1\} & \text{if } c > 0, \end{cases}$$

so $\operatorname{gph}(\partial h)$ is not closed. On the other hand, the subdifferentials of lower semicontinuous *convex* functions are closed.

Semialgebraic sets and functions offer a rich class of examples for this decomposition approach. A subset of $\mathbb{R}^n$ is *semialgebraic* if it can be expressed as a finite union of sets, each of which is defined by finitely many polynomial inequalities. The semialgebraic property is a powerful tool, in part because it is preserved under linear images (an application of the Tarski–Seidenberg quantifier elimination theorem). Consequently, semialgebraic sets are often easy to recognize without knowing an explicit representation in terms of polynomials. A function is *semialgebraic* when it has a semialgebraic graph. In general, for any semialgebraic function h, a standard quantifier elimination argument shows that the set $\operatorname{gph}(\partial h)$ is semialgebraic. If this set is also closed, then it stratifies into a finite union of smooth manifolds with boundaries. A good recent reference on semialgebraic geometry is [1]. In concrete cases, a decomposition may be reasonably straightforward.

We end this section with two examples.

Example 4.4 (Euclidean norm). The graph of the subdifferential of the Euclidean norm on $\mathbb{R}^n$ decomposes into the union of the following two closed sets:

$$\{(0,v) \colon |v| \le 1\} \quad \text{and} \quad \left\{\left(c, \frac{1}{|c|}c\right) \colon c \ne 0\right\} \cup \{(0,v) \colon |v| = 1\}.$$

Example 4.5 (maximum eigenvalue). Consider the maximum eigenvalue function $\lambda_{\max}$ on the Euclidean space $\mathbf{S}^k$ of k-by-k symmetric matrices (with the inner product $\langle X, Y\rangle = \mathrm{trace}(XY)$). In this space, the following sets are closed:

$$\mathbf{S}^k_r = \{\, Y \in \mathbf{S}^k \colon Y \text{ has rank} \le r\} \quad (r = 0, 1, \ldots, k),$$
$$_m\mathbf{S}^k = \{X \in \mathbf{S}^k \colon \lambda_{\max}(X) \text{ has multiplicity} \ge m\} \quad (m = 1, 2, \ldots, k).$$

Trivially we can decompose the graph $\mathrm{gph}(\partial\lambda_{\max})$ into its intersection with each of the sets $_m\mathbf{S}^k \times \mathbf{S}^k_r$. However, we can simplify, since it is well known (see [10], for example) that $\partial\lambda_{\max}(X)$ consists of matrices of rank no more than the multiplicity of $\lambda_{\max}(X)$. Hence we can decompose the graph into the union of the sets

$$G_{m,r} = \mathrm{gph}(\partial\lambda_{\max}) \cap (_m\mathbf{S}^k \times \mathbf{S}^k_r)(1 \le r \le m \le k).$$

To apply the theory we have developed, we need to measure the distance from any given pair (X, Y) in the graph to each of the sets $G_{m,r}$. This is straightforward, as follows. A standard characterization of $\partial\lambda_{\max}$ [10] shows that there must exist an orthogonal matrix U, a vector $x \in \mathbb{R}^k$ with nonincreasing components, and a vector $y \in \mathbb{R}^k_+$ satisfying $\sum_i y_i = 1$ and $y_i = 0$ for all indices $i > p$, where p is the multiplicity of the largest component of x such that the following simultaneous spectral decomposition holds: $X = U^T(\mathrm{Diag}\, x)U$ and $Y = U^T(\mathrm{Diag}\, y)U$. Now define a vector $\tilde{x} \in \mathbb{R}^k$ by replacing the first m components of x by their mean. (Notice that the components of $\tilde{x}$ are then still in nonincreasing order, and the largest component has multiplicity at least p.) Define a vector $\tilde{y} \in \mathbb{R}^k$ by setting all but the largest r components of y to zero and then rescaling the resulting vector to ensure its components sum to one. (Notice that $\tilde{y}_i = 0$ for all indices $i > p$.) Finally, define matrices $\tilde{X} = U^T(\mathrm{Diag}\,\tilde{x})U$ and $\tilde{Y} = U^T(\mathrm{Diag}\,\tilde{y})U$. Then, by the same subdifferential characterization, we have $\tilde{Y} \in \partial\lambda_{\max}(\tilde{X})$, so, in fact, $(\tilde{X}, \tilde{Y}) \in G_{m,r}$. Hence the distance from (X, Y) to $G_{m,r}$ is at most $\sqrt{|x - \tilde{x}|^2 Q + |y - \tilde{y}|^2}$. In fact, this easily computable estimate is exact, since it is well known that $\tilde{Y}$ is a closest matrix to Y in the set $\mathbf{S}^k_r$ and, by [12, Example A.4], $\tilde{X}$ is a closest matrix to X in the set $_m\mathbf{S}^k$.

Now consider a nonlinear eigenvalue optimization problem of the form

$$\min_x \lambda_{\max}(c(x)).$$

Combining the distance computation above with a result such as Corollary 3.5 gives an approach to bounding the multiplicity, at optimality, of the largest eigenvalue $\lambda_{\max}(\bar{c})$, along with the rank of the corresponding multiplier matrix.

5. Nonlinear programming revisited.

Our intent in this work is not to develop fresh results for classical nonlinear programming, but rather to generalize previous results in an intuitive framework suggesting new applications. Nonetheless, it is worthwhile to verify how our main result recaptures the classical case that we discussed in section 2 and Example 4.3. The following result is not hard to prove directly from the Mangasarian–Fromovitz condition, but we present it as an illustration of our technique.

Corollary 5.1. *Consider a critical point $\bar{x} \in \mathbb{R}^n$ for the nonlinear program* (NLP)*, where the objective function and each of the constraint functions are all continuously differentiable. Suppose the Mangasarian–Fromovitz condition (Assumption* 2*) holds.*

Then there exists a number $\bar{\epsilon} > 0$ such that if the sequences $x^r \in \mathbb{R}^n$, $\lambda^r \in \mathbb{R}^s$, $\phi^r \in \mathbb{R}^t_+$ (for $r = 1, 2, 3, \ldots$) satisfy

$$x^r \to \bar{x}, \tag{5.1a}$$

$$\nabla f(x^r) + \sum_{i=1}^{s} \lambda_i^r \nabla p_i(x^r) + \sum_{j=1}^{t} \phi_j^r \nabla q_j(x^r) \to 0, \tag{5.1b}$$

$$\min\{q_j(x^r) : \phi_j^r > 0, j = 1, 2, \ldots, t\} \to 0 \tag{5.1c}$$

(interpreting $\min \varnothing = 0$*), then any index set $J \subset \{1, 2, \ldots, t\}$ satisfying*

$$q_j(x^r) > -\bar{\epsilon} \ (for\ all\ j \in J) \quad and \quad \phi_j^r < \bar{\epsilon} \ (for\ all\ j \notin J)$$

for infinitely many r is sufficient for $\bar{x}$.

Proof. By applying Corollary 3.3 to the problem discussed in section 2 with the graph decomposition described in Example 4.3 and simplifying slightly, we deduce the existence of a number $\epsilon > 0$ such that if the sequences $x^r \in \mathbb{R}^n$, $w^r \in \mathbb{R}^t_-$, $\lambda^r \in \mathbb{R}^s$, $\phi^r \in \mathbb{R}^t_+$ satisfy $\langle w^r, \phi^r \rangle = 0$ for all $r = 1, 2, 3, \ldots$ and

$$x^r \to \bar{x},$$
$$w^r \to q(\bar{x}),$$
$$\nabla f(x^r) + \sum_{i=1}^{s} \lambda_i^r \nabla p_i(x^r) + \sum_{j=1}^{t} \phi_j^r \nabla q_j(x^r) \to 0,$$

then any index set $J \subset \{1, 2, \ldots, t\}$ satisfying $\mathrm{dist}((w^r, \phi^r), M^J) < \epsilon$ for infinitely many r, where

$$M^J = \{(w, \phi) \in \mathbb{R}^t_+ \times \mathbb{R}^t_- : w_j = 0 \ (j \in J), \ \phi_j = 0 \ (j \notin J)\},$$

is sufficient for $\bar{x}$. (To be precise, using the notation (2.1), we set $c^r = (f(\bar{x}), 0, w^r)$ and $v^r = (1, \lambda^r, \phi^r)$, and we note that $\mathrm{dist}((c^r, w^r), G^J) = \mathrm{dist}((w^r, \phi^r), M^J)$, where G^J is defined in (4.5).) Assuming the hypotheses of the theorem, and setting $\bar{\epsilon} = \epsilon/\sqrt{t}$, we show that this observation can be applied with the sequence $w^r \in \mathbb{R}^t_-$ defined by

$$w_j^r = \begin{cases} 0 & \text{if } \phi_j^r > 0, \\ \min\{q_j(x^r), 0\} & \text{if } \phi_j^r = 0. \end{cases}$$

Clearly each $\langle w^r, \phi^r \rangle = 0$ for all $r = 1, 2, 3, \ldots$, so we only need to check $w^r \to q(\bar{x})$. If this fails, there is an index j, a number $\delta > 0$, and a subsequence R of $\mathbf{N}$ such that $|w_j^r - q_j(\bar{x})| > \delta$ for all $r \in R$ and either $\phi_j^r = 0$ for all $r \in R$ or $\phi_j^r > 0$ for all $r \in R$. Notice $q_j(x^r) \to q(\bar{x}) \le 0$. Hence in the first case we obtain the contradiction

$$w_j^r = \min\{q_j(x^r), 0\} \to \min\{q_j(\bar{x}), 0\} = q_j(\bar{x}),$$

while in the second case we have $w_j^r = 0$, and hence $q_j(\bar{x}) < -\delta$. Thus, (5.1c) is violated, giving a contradiction.

By considering the point $(\hat{w}, \hat{\phi}) \in M^J$ defined by

$$\hat{w}_j^r = \begin{cases} w_j^r & \text{if } j \notin J, \\ 0 & \text{if } j \in J, \end{cases} \qquad \hat{\phi}_j^r = \begin{cases} \phi_j^r & \text{if } j \in J, \\ 0 & \text{if } j \notin J, \end{cases}$$

we have that

$$\operatorname{dist}^2((w^r,\phi^r),M^J) \le \|(w^r,\phi^r)-(\hat{w},\hat{\phi})\|^2 = \sum_{j\in J}(w_j^r)^2 + \sum_{j\notin J}(\phi_j^r)^2 < t\bar{\epsilon}^2,$$

where the last inequality follows from $w_j^r \le 0$, $\phi_j^r \ge 0$, and

$$j \in J \Rightarrow q_j(x^r) > -\bar{\epsilon} \Rightarrow w_j^r > -\bar{\epsilon}, \qquad j \notin J \Rightarrow \phi_j^r < \bar{\epsilon}.$$

From $\bar{\epsilon} = \epsilon/\sqrt{t}$, we thus have $\operatorname{dist}((c^r,v^r),G^J) = \operatorname{dist}((w^r,\phi^r),M^J) < \epsilon$, completing the proof. □

6. Partial smoothness. We next observe a connection between the decomposition ideas we have introduced and the notion of "partial smoothness" [11]. This notion abstracts and generalizes a variety of earlier work on the identification problem, including, in particular, [3], [2], [4]. For simplicity, we restrict the discussion in this section to the convex case, although extensions are possible. In the convex case, partial smoothness is equivalent to the idea introduced in [21] of an "identifiable surface."

Consider a lower semicontinuous convex function $h\colon \mathbb{R}^m \to \bar{\mathbb{R}}$, a point $\bar{c} \in \mathbb{R}^m$, any vector $\bar{v} \in \operatorname{ri}\partial h(\bar{c})$, and a set $\mathcal{M}$ containing $\bar{c}$. We call h *partly smooth* at $\bar{c}$ relative to $\mathcal{M}$ when the following properties hold:

(i) $\mathcal{M}$ is a manifold around $\bar{c}$;
(ii) the restricted function $h|_{\mathcal{M}}$ is C^2;
(iii) the subdifferential mapping ∂h is continuous at $\bar{c}$ when restricted to $\mathcal{M}$;
(iv) the affine span of $\partial h(\bar{c}) - \bar{v}$ is $N_{\mathcal{M}}(\bar{c})$.

We assume that these properties hold and, in addition, that $\mathcal{M}$ is closed.

First note that the graph of the subdifferential ∂h is the union of the following two sets:

$$G^1 = \{(c,v)\colon c \in \mathcal{M}, v \in \partial h(c)\}, \qquad G^2 = \operatorname{cl}\{(c,v)\colon c \notin \mathcal{M}, v \in \partial h(c)\}.$$

We have some simple observations.

LEMMA 6.1. *The set G^1 is closed.*

Proof. As is well known (see, for example, [19, Theorem 24.4]), since h is convex and lower semicontinuous, $\operatorname{gph}(\partial h)$ is closed. Indeed we can write it as the lower level set of a lower semicontinuous function:

$$\operatorname{gph}(\partial h) = \{(c,v)\colon h(c) + h^*(v) - \langle c, v\rangle \le 0\},$$

where h^* denotes the Fenchel conjugate of h. Since $G^1 = \operatorname{gph}(\partial h) \cap (\mathcal{M} \times \mathbb{R}^m)$, and since $\mathcal{M}$ is closed by assumption, the result follows. □

LEMMA 6.2. $(\bar{c},\bar{v}) \notin G^2$.

Proof. If this property fails, then there is a sequence of points $c^r \notin \mathcal{M}$ $(r = 1,2,\ldots)$ approaching $\bar{c}$ and a corresponding sequence of subgradients $v^r \in \partial h(c^r)$ approaching $\bar{v}$. A standard subdifferential continuity argument can now be used to show that $h(c^r) \to h(\bar{c})$. To be precise, it follows from lower semicontinuity of h (and of h^*) that $\liminf_r h(c^r) \ge h(\bar{c})$ and $\liminf_r h^*(v^r) \ge h^*(\bar{v})$. Thus,

$$\begin{aligned}\lim_r \sup h(c^r) = \lim_r \sup\left(\langle c^r, v^r\rangle - h^*(v^r)\right) = \langle \bar{c}, \bar{v}\rangle - \lim_r \inf h^*(v^r)\\ \le \langle \bar{c}, \bar{v}\rangle - h^*(\bar{v}) = h(\bar{c}) \le \lim_r \inf h(c^r),\end{aligned}$$

where the first and last equalities follow from [20, Proposition 11.3]. An easy modification of [8, Theorem 5.3] (see [13, Theorem 6.11]) implies the contradiction $c^r \in \mathcal{M}$ for all large r. ∎

We can now interpret how partial smoothness leads to identification in the light of our main result.

Corollary 6.3. *Suppose Assumption* 1 *holds. Suppose that the critical point* $\bar{x}$ *has a unique multiplier vector* $\bar{v}$ *and that* $\bar{v} \in \operatorname{ri} \partial h(c(\bar{x}))$. *Finally, assume that* h *is convex and also partly smooth at the point* $c(\bar{x})$ *relative to a closed set* $\mathcal{M} \subset \mathbb{R}^m$. *Then any sufficiently accurate solution of the criticality conditions near* $\bar{x}$ *must identify the set* $\mathcal{M}$. *More precisely, for any sequence of points* $x^r \in \mathbb{R}^n$, *vectors* $c^r \in \mathbb{R}^m$, *and subgradients* $v^r \in \partial h(c^r)$ *(for* $r = 1, 2, \dots$*), satisfying*

$$x^r \to \bar{x}, \quad |c^r - c(x^r)| \to 0, \quad h(c^r) \to h(c(\bar{x})), \quad \nabla c(x^r)^* v^r \to 0$$

as $r \to \infty$, *we must have* $c^r \in \mathcal{M}$ *for all sufficiently large* r.

Proof. By Lemma 6.1, $\operatorname{gph}(\partial h) = G^1 \cup G^2$ is a decomposition into closed sets. By Lemma 6.2, the set G^2 is not actively sufficient. Corollary 3.5 then implies that there is $\hat{\epsilon} > 0$ such that $\operatorname{dist}((c^r, v^r), G^2) > \hat{\epsilon}$ for all r sufficiently large. The result follows. ∎

For a variety of algorithmic approaches to identification in the partly smooth setting, see [7] and [9]. These works discuss, in particular, conditions under which gradient projection schemes, Newton-like methods, and proximal point methods identify the set $\mathcal{M}$ in the result above. They also discuss how a simple proximal-type subproblem can accomplish the same goal in more general algorithmic contexts.

7. Identifying activity via a proximal subproblem. In this section we consider the question of whether closed sets G that are actively sufficient at a solution $\bar{x}$ of the composite minimization problem (1.1) can be identified from a nearby point x by solving the subproblem

$$\min_d h_{x,\mu}(d) := h(c(x) + \nabla c(x) d) + \frac{\mu}{2}|d|^2. \tag{7.1}$$

Properties of local solutions of this subproblem and of a first-order algorithm based on it have been analyzed by the authors in [13]. In that work, we gave conditions guaranteeing, in particular, that if the function h is partly smooth relative to some manifold $\mathcal{M}$ containing $c(\bar{x})$, where $\bar{x}$ is the critical point, then the subproblem (7.1) "identifies" $\mathcal{M}$; that is, nearby local minimizers must lie on $\mathcal{M}$.

The identification result from [13] requires a rather strong regularity condition at the critical point $\bar{x}$. When applied to the case of classical nonlinear programming described above, this condition reduces to the linear independence constraint qualification, in particular, always implying uniqueness of the multiplier vector. In the simplest case, when, in addition, strict complementarity holds, there is a unique sufficient index set (in the terminology of section 5), and the identification result Corollary 6.3 applies.

By contrast, in this section, we pursue more general identification results, needing only the transversality condition (1.3). Certain additional assumptions on the function h are required, whose purpose is essentially to ensure that the solution of (7.1) is well behaved.

We start with some technical results from [13], and then we state our main result. In what follows, we use the idea of "prox-regularity" of a function h. We refer the reader to [20] for the rather technical definition. Here we simply observe that an intuitive and broadly applicable sufficient condition for prox-regularity of h at a point $\bar{c}$ with finite value is the property that every point sufficiently close to the point $(\bar{c}, h(\bar{c}))$ has a unique nearest point in the epigraph of h. For more details, see [17, Theorem 3.5] and [18, Theorem 1.3]. In particular, lower semicontinuous convex functions are everywhere prox-regular, as are sums of continuous convex functions and $\mathcal{C}^2$ functions.

For the results that follow, we need to strengthen our underlying Assumption 1 as follows.

Assumption 3. In addition to Assumption 1, the function c is $\mathcal{C}^2$ around the critical point $\bar{x}$, and the function h is prox-regular at the point $\bar{c} = c(\bar{x})$.

The following result is a restatement of [13, Theorem 6.5]. It concerns existence of local solutions to (7.1) with nice properties.

THEOREM 7.1. *Suppose Assumption* 3 *holds. Then there exist numbers $\bar{\mu} \geq 0$, $\delta > 0$, and $k > 0$ and a mapping $d\colon B_\delta(\bar{x}) \times (\bar{\mu}, \infty) \to \mathbb{R}^n$ such that the following properties hold.*

(a) *For all points $x \in B_\delta(\bar{x})$ and all scalars $\mu > \bar{\mu}$, the point $d(x, \mu)$ is a local minimizer of the subproblem* (7.1) *and, moreover, satisfies $|d(x,\mu)| \leq k|x - \bar{x}|$.*

(b) *Given any sequences of points $x^r \to \bar{x}$ and scalars $\mu_r > \bar{\mu}$, if either $h(c(x^r)) \to h(\bar{c})$ or $\mu_r|x^r - \bar{x}|^2 \to 0$, then*

$$h(c(x^r) + \nabla c(x^r) d(x^r, \mu_r)) \to h(\bar{c}). \tag{7.2}$$

(c) *When h is convex and lower semicontinuous, the results of parts* (a) *and* (b) *hold with $\bar{\mu} = 0$.*

The next result is a slightly abbreviated version of [13, Lemma 6.7].

LEMMA 7.2. *Suppose Assumption* 3 *holds. Then for any sequences $\mu_r > 0$ and $x^r \to \bar{x}$ such that $\mu_r|x^r - \bar{x}| \to 0$, and any corresponding sequence of critical points d^r for the subproblem* (7.1) *that satisfy the conditions*

$$d^r = O(|x^r - \bar{x}|) \quad \textit{and} \quad h(c(x^r) + \nabla c(x^r) d^r) \to h(\bar{c}), \tag{7.3}$$

there exists a bounded sequence of vectors v^r that satisfy

$$0 = \nabla c(x^r)^* v^r + \mu_r d^r, \tag{7.4a}$$

$$v^r \in \partial h(c(x^r) + \nabla c(x^r) d^r). \tag{7.4b}$$

If we assume, in addition, that $\mu_r > \bar{\mu}$, where $\bar{\mu}$ is defined in Theorem 7.1, the vectors $d^r := d(x^r, \mu_r)$ satisfy the properties (7.3), and hence the results of Lemma 7.2 apply.

We now prove the main result of this section.

THEOREM 7.3. *Suppose Assumption* 3 *holds, and consider a closed set $G \subset \mathrm{gph}(\partial h)$. Consider any sequences of scalars $\mu_r > 0$ and points $x^r \to \bar{x}$ satisfying the condition $\mu_r|x^r - \bar{x}| \to 0$, and let d^r be any corresponding sequence of critical points of the subproblem* (7.1) *satisfying* (7.3)*. Consider any corresponding sequence of vectors v^r satisfying the conditions* (7.4) *for which*

$$\mathrm{dist}((c(x^r) + \nabla c(x^r) d^r, v^r), G) \to 0. \tag{7.5}$$

Then G is an actively sufficient set at $\bar{x}$.

Proof. We apply Corollary 3.5 with $\mathcal{G} = \{G\}$ and $c^r := c(x^r) + \nabla c(x^r)d^r$. Because of the various properties of x^r, d^r, v^r, and μ_r, from Theorem 7.1 and Lemma 7.2, we have the following estimates:

$$\begin{aligned} x^r - \bar{x} &\to 0, \\ v^r &\in \partial h(c^r), \\ |c^r - c(x^r)| = |\nabla c(x^r)d^r| = O(|d^r|) = O(|x^r - \bar{x}|) &\to 0, \\ h(c^r) - h(c(\bar{x})) &\to 0, \\ |\nabla c(x^r)^* v^r| = \mu_r |d^r| = \mu_r O(|x^r - \bar{x}|) &\to 0, \\ \operatorname{dist}((c^r, v^r), G) &\to 0. \end{aligned}$$

The result follows. □

Note again that Theorem 7.1 and Lemma 7.2 show that vectors d^r satisfying the conditions of Theorem 7.3 can be obtained when $\mu_r > \bar{\mu}$, and we can take $\bar{\mu} = 0$ when h is convex and lower semicontinuous.

As we have seen, in particular in the case of classical nonlinear programming, we typically have in mind some "natural" decomposition of the subdifferential graph $\operatorname{gph}(\partial h)$ into the union of a finite family $\mathcal{G}$ of closed subsets. We then somehow generate sequences, μ_r, x^r, d^r, and v^r of the type specified in the theorem, and thereby try to identify actively sufficient sets in $\mathcal{G}$, preferring smaller sets since the corresponding restricted criticality system is then more refined. Since $\mathcal{G}$ is a finite family, Theorem 7.3 guarantees that we must identify at least one actively sufficient set in this way—but possibly not *all* actively sufficient sets. In other words, a sequence of iterates generated by the algorithm based on (7.1) and corresponding multiplier vectors may "reveal" some of the actively sufficient sets but not others. We illustrate this point with an example based on a degenerate nonlinear optimization problem in two variables.

Example 7.1. Consider the map $c: \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$c(x) = \begin{bmatrix} -x_1 \\ x_1^2 + x_2^2 - 1 \\ (x_1+1)^2 + x_2^2 - 4 \end{bmatrix}$$

and the function $h: \mathbb{R}^3 \to \bar{\mathbb{R}}$ defined by

$$h(c) = \begin{cases} c_1 & \text{if } c_2, c_3 \le 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Minimizing the composite function $h \circ c$ thus amounts to maximizing x_1 over the set in $\mathbb{R}^2$ defined by the constraints $|x| \le 1$ and $|x - (-1,0)^T| \le 2$. The unique minimizer of $h \circ c$ is the point $\bar{x} = (1,0)^T$, at which $c(\bar{x}) = (-1,0,0)^T$. The set of multiplier vectors is

$$\partial h(c(\bar{x})) \cap N(\nabla(c(\bar{x}))^*) = \left\{ \alpha \left(1, \frac{1}{2}, 0\right)^T + (1-\alpha)\left(1, 0, \frac{1}{4}\right)^T : \alpha \in [0,1] \right\}.$$

We can decompose $\operatorname{gph}(\partial h)$ as the union of the following four closed sets:

$$\begin{aligned}
G^1 &= \{(c_1, c_2, c_3, 1, 0, 0) \colon c_2 \le 0, c_3 \le 0\},\\
G^2 &= \{(c_1, 0, c_3, 1, v_2, 0) \colon v_2 \ge 0, c_3 \le 0\},\\
G^3 &= \{(c_1, c_2, 0, 1, 0, v_3) \colon c_2 \le 0, v_3 \ge 0\},\\
G^4 &= \{(c_1, 0, 0, 1, v_2, v_3) \colon v_2 \ge 0, v_3 \ge 0\}.
\end{aligned}$$

(We can refine further, but this suffices for our present purpose.) In this decomposition, the actively sufficient subsets are G^2, G^3, G^4.

The subproblem (7.1), applied from some point $x = (x_1, 0)^T$ with x_1 close to 1, reduces to

$$\begin{aligned}
&\text{minimize } -d_1 + \frac{\mu}{2}(d_1^2 + d_2^2)\\
&\text{subject to } d_1 \le \frac{1}{2x_1} - \frac{x_1}{2},\\
&\qquad d_1 \le \frac{2}{x_1+1} - \frac{x_1+1}{2},\\
&\qquad d \in \mathbb{R}^2.
\end{aligned}$$

If $x_1 = 1 - \epsilon$ for some small ϵ (not necessarily positive), the constraints reduce to

$$\begin{aligned}
d_1 &\le \epsilon + \frac{1}{2}\epsilon^2 + O(\epsilon^3),\\
d_1 &\le \epsilon + \frac{1}{4}\epsilon^2 + O(\epsilon^3).
\end{aligned}$$

Providing $\epsilon \ll \frac{1}{\mu}$, the solution of the subproblem has $d_1 \approx \epsilon + \frac{1}{4}\epsilon^2$ and $d_2 = 0$. The corresponding linearized values of c_2 and c_3 are

$$c_2(x) + \nabla c_2(x)^T d \approx \frac{1}{4}\epsilon^2, \qquad c_3(x) + \nabla c_3(x)^T d = 0,$$

and the corresponding multiplier vector is $v \approx (1, 0, \frac{1}{4})^T$. Thus this iterate "reveals" the actively sufficient sets G^3 and G^4, but not G^2.

Subsequent iterates generated by this scheme have the identical form $(1-\epsilon, 0)^T$ with successively smaller values of ϵ, so the sequence satisfies the property (7.5) only for $G = G^3$ and $G = G^4$, but not for $G = G^2$.

Consider again the nonlinear programming formulation of sections 2 and 5. In that framework, for a given point $x \in \mathbb{R}^n$, the proximal subproblem (7.1) is the following quadratic program:

$$\text{minimize } f(x) + \nabla f(x)^T d + \frac{\mu}{2}|d|^2 \tag{7.6a}$$

$$\text{subject to } p_i(x) + \nabla p_i(x)^T d = 0 \quad (i = 1, 2, \dots, s), \tag{7.6b}$$

$$q_j(x) + \nabla q_j(x)^T d \le 0 \quad (j = 1, 2, \dots, t), \tag{7.6c}$$

$$d \in \mathbb{R}^n. \tag{7.6d}$$

We derive the following corollary as a simple application of Theorem 7.3.

Corollary 7.4. *Consider the nonlinear program* (NLP)*, where the functions f, p_i $(i = 1, 2, \ldots, s)$, and q_j $(j = 1, 2, \ldots, t)$ are all $\mathcal{C}^2$ around the critical point $\bar{x}$, and suppose that the Mangasarian–Fromovitz constraint qualification, Assumption* 2*, holds. Consider sequences of scalars $\mu_r > 0$ and points $x^r \to \bar{x}$ satisfying $\mu_r|x^r - \bar{x}| \to 0$, let d^r be the corresponding (unique) solution of* (7.6)*, and consider an additional sequence of nonnegative tolerances $\epsilon_r \to 0$. Then for all sufficiently large r, the index set $J(r) \subset \{1, 2, \ldots, t\}$ defined by*

$$J(r) := \{j : q_j(x^r) + \nabla q_j(x^r)^T d^r \geq -\epsilon_r\} \tag{7.7}$$

is sufficient for $\bar{x}$.

Proof. Let $J \subset \{1, 2, \ldots, t\}$ be any set such that $J(r) = J$ for infinitely many r. The result is proved if we can show that J is actively sufficient. We can assume without loss of generality that $J(r) \equiv J$. Noting convexity of the function h defined in (2.1b) and the equivalence of the transversality condition (1.3) and Assumption 2, we have from Theorem 7.1 that the unique solution of the subproblem (7.6) satisfies $d^r = O(|x^r - \bar{x}|)$ and

$$h(c(x^r) + \nabla c(x^r) d^r) = f(x^r) + \nabla f(x^r)^T d^r \to f(\bar{x}) = h(c(\bar{x})).$$

Moreover, the distance between the point

$$((f(x^r) + \nabla f(x^r)^T d^r, p(x^r) + \nabla p(x^r) d^r, q(x^r) + \nabla q(x^r) d^r), (1, \lambda^r, \phi^r))$$

and the set G^J defined in (4.5) approaches zero, where λ^r and ϕ^r are the multipliers for the linear constraints in the subproblem (7.6). (This claim follows from (7.7) and the fact that $q_j(x^r) + \nabla q_j(x^r)^T d^r < -\epsilon_r < 0$ for all $j \notin J$ so that $\phi_j^r = 0$.) We conclude from Theorem 7.3 that G^J is an actively sufficient set at $\bar{x}$, so the index set J is sufficient. ∎

Similar results hold for a nonsmooth penalty formulation of the nonlinear program (NLP). For example, the ℓ_1-penalty formulation corresponds to the function h defined as follows:

$$h(u, v, w) = u + \nu\left(\sum_{i=1}^{s} |v_i| + \sum_{j=1}^{t} \max(w_j, 0)\right).$$

The corresponding proximal subproblem (7.1) at some given point $x \in \mathbb{R}^n$ is as follows:

$$\min_{d \in \mathbb{R}^n} f(x) + \nabla f(x)^T d + \frac{\mu}{2}|d|^2 + \nu\left(\sum_{i=1}^{s} |p_i(x) + \nabla p_i(x)^T d| + \sum_{j=1}^{s} \max(q_j(x) + \nabla q_j(x)^T d, 0)\right)$$

for a given penalty parameter $\nu > 0$. A result similar to Corollary 7.4 for this formulation would lead to an identification result like Theorem 3.2 of [16], provided that ν is large enough to bound the ℓ_∞ norm of all multipliers that satisfy the stationarity conditions for (NLP). A notable difference, however, is that [16, Theorem 3.2] uses a trust-region of the form $\|d\|_\infty \leq \Delta$ to restrict the size of the solution d, whereas this subproblem uses the prox term $\frac{\mu}{2}|d|^2$. Although the use of an ℓ_∞ trust-region allows the subproblem to be formulated as a linear program, the radius Δ must satisfy certain conditions, not easily verified, for the identification result to hold. By contrast, there are no requirements on μ in the subproblems above, beyond positivity.

A possible extension we do not pursue here allows an extra term $\frac{1}{2}\langle d, Bd\rangle$ for some monotone operator B, in addition to the prox term $\frac{\mu}{2}|d|^2$. This generalization allows SQP-type subproblems to be considered, which potentially could be useful in analyzing algorithms combining identification and second-order steps into a single iteration (as happens with traditional SQP methods).

8. Discussion. We have shown how certain interesting subsets of $\text{gph}(\partial h)$ can be identified from a sequence of points $x_r \to \bar{x}$ (where $\bar{x}$ is a critical point of (1.1)) by making use of information about ∇c and ∂h at x_r and $c(x_r)$, respectively. These subsets are potentially useful in algorithms because they allow an algorithm for solving (1.1) to eventually focus its attention on a reduced space, often leading to significant savings in computational costs. The economics of different strategies depend critically on the properties of the functions, but an obvious idea would be to base the early stages of an algorithm on "first-order" strategy (like (7.6)), switching to steps that make more explicit use of second-order information when the reduced space is identified with some confidence. This strategy may save unnecessary evaluation and manipulation of full-space second-order information remote from the solution, where it is not very helpful. It may also save unnecessary evaluation of full-space first-order information in the final stages of a method—another source of savings that is significant in some applications.

Any algorithm should, of course, be able to recover from an "incorrect" subset identification. The simplest strategy would be to return to the "outer loop," discarding the steps that were taken on the basis of the incorrect identification. Often, however, the "inner loop" method itself will be able to make minor adjustments to the identified set. In the case of nonlinear programming, for example, a few additional pivots might be required during the stages of an SQP method for linear programming to correctly classify the last few active and inactive constraints.

We expect that in many practical situations, heuristics rather than rigorous procedures would be used to identify the active subsets. The results of this paper provide theoretical support and motivation for such heuristics and procedures.

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