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Critical points of simple functions

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Among various notions of critical points available for nonsmooth functions, the approach via the 'weak slope' has considerable appeal. This property is less restrictive than that based on the 'strong' slope of DeGiorgi–Marino–Tosques, but remains purely metric in nature. Within variational analysis, this class of critical points is intermediate between those associated with the Clarke and the limiting subdifferentials. However, recognizing such points for concrete functions seems challenging. We present a basic topological characterization for the simplest nontrivial case: piecewise affine (and, more generally, 'definable') functions of two variables.

Keywords: nonsmooth critical point; weak slope; subdifferential

AMS 2000 Subject Classifications: 49J52; 58E05

1. Nonsmooth critical points

Throughout this article, we consider a continuous function $f: \mathbf{R}^n \to \mathbf{R}$. We begin by considering some purely metric notions of critical points.

We denote the closed ball of radius $\delta > 0$ centered at a point $x \in \mathbf{R}^n$ by $B_\delta(x)$. We recall the following definitions from Degiovanni–Marzocchi [11], Katriel [15] and Ioffe–Schwartzman [14].

Definition 1.1 We call the point x *Morse regular from below* for the function f if, for some numbers $\delta, \sigma > 0$, there is a continuous function

$$\phi : B_\delta(x) \times [0, \delta] \to \mathbf{R}^n$$

such that all points $u \in B_\delta(x)$ and $t \in [0, \delta]$ satisfy the inequality

$$f(\phi(u, t)) \le f(u) - \sigma t.$$

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If there is a number $\kappa > 0$ and such a function ϕ that also satisfies the inequality

$$\|\phi(u, t) - u\| \leq \kappa t,$$

then we call x 'deformationally regular from below'. (We can arbitrarily fix one of the numbers σ or κ, so we usually set $\kappa = 1$.) Finally, we call x 'deformationally critical from below' if it is not deformationally regular from below. The supremum of the possible ratios σ/κ is called the 'weak slope' of f at x.

Clearly, the set of points that are Morse or deformationally regular from below is open. It is also clear that deformational regularity implies Morse regularity, and so, any Morse critical point is deformationally critical. An example of a deformationally critical point that is not Morse critical is given by the function $f(t) = t^3$.

The function ϕ in the above definition is called a 'deformation'. Deformations are among the principal technical tools in classical critical point theory. In the classical smooth setting, the existence of a suitable deformation in a neighbourhood of a point at which the function has a nonzero derivative is one of the central technical facts. For an early use of the deformation idea for nonsmooth functions, see [8,14]. Subsequent developments along this line include [7].

We note that the property of deformational regularity is unaffected by well-behaved local homeomorphisms. Specifically, we have the following result.

Proposition 1.2 (regularity and homeomorphisms) *Suppose that the point $x \in \mathbf{R}^n$ is Morse (respectively deformationally) regular from below for the function $f: \mathbf{R}^n \to \mathbf{R}$, and that the map $G: \mathbf{R}^n \to \mathbf{R}^n$ is a homeomorphism (respectively, a Lipschitz homeomorphism) of a neighbourhood of x. Then, the point $G(x)$ is Morse (respectively deformationally) regular from below for the function $f \circ G^{-1}$.*

Proof Assume the existence of a deformation ϕ as in Definition 1.1. We construct a new deformation ψ on a suitable domain by setting

$$\psi(v, t) = G(\phi(G^{-1}(v), t)).$$

Then we have

$$f(G^{-1}(v)) - f(G^{-1}((\psi(v, t)))) = f(G^{-1}(v)) - f(\phi(G^{-1}(v), t)) \geq \sigma t.$$

If f is deformationally regular and G is a Lipschitz homeomorphism with the local Lipschitz constant κ', then we also have

$$\begin{aligned} \|\psi(v, t) - v\| &= \|G(\phi(G^{-1}(v), t)) - G(G^{-1}(v))\| \\ &\leq \kappa \|\phi(G^{-1}(v), t) - G^{-1}(v)\| \\ &\leq \kappa\kappa' t, \end{aligned}$$

completing the proof. ■

By contrast with this approach, let us call the point x 'strongly critical from below' if

$$f(u) - f(x) \geq o(\|u - x\|) \quad \text{for } u \text{ near } x.$$

In the terminology of De-Giorgi–Marino–Tosques [12], f has (strong) slope zero at x.

Clearly, any local minimizer is strongly critical. Less obvious is the following fact.

Proposition 1.3 (local maximizers) *Any local maximizer is Morse critical from below.*

Proof Suppose the point x is a local maximizer and yet is Morse regular from below. Using the notation of Definition 1.1, we can suppose without loss of generality that $x=0$ and $f(u)\leq f(0)$ whenever $\|u\|\leq\delta$.

Suppose $\|u\|\leq\delta$ and $\phi(u,\,\delta-\|u\|)=0$. Then we have

$$f(0)=f(\phi(u,\delta-\|u\|))\leq f(u)-\sigma(\delta-\|u\|)\leq f(u)\leq f(0),$$

so equality holds throughout. In particular, we deduce $\|u\|=\delta$. But this implies the contradiction $0=\phi(u,\,0)=u$.

We have therefore shown that $\phi(u,\,\delta-\|u\|)\neq 0$ whenever $\|u\|\leq\delta$. But now the map

$$u\mapsto\delta\frac{\phi(u,\delta-\|u\|)}{\|\phi(u,\delta-\|u\|)\|}$$

is a retraction from $B_\delta(0)$ to its boundary, contradicting the Brouwer fixed point theorem. ■

For continuously differentiable functions f, it is easy to see that all the above notions of critical point correspond to $f'(x)=0$. For nonsmooth f, various definitions of critical point compete, depending on the context: we next summarize two more of the most important notions.

We call a vector $y\in\mathbf{R}^n$ a 'Fréchet subgradient' of f at the point x if

$$f(u)-f(x)\geq y^T(u-x)+o(\|u-x\|)\quad\text{for } u \text{ near } x.$$

Thus, x is strongly critical exactly when zero is a Fréchet subgradient there. We call y a 'limiting subgradient' if there is a sequence of points $u_r\in\mathbf{R}^n$ approaching x and a sequence of Fréchet subgradients y_r at u_r approaching y. We denote the set of such limiting subgradients by $\partial^L f(x)$, and we call x 'limiting critical' at x when $0\in\partial^L f(x)$.

The function f is 'locally Lipschitz' around the point x when, for some $\kappa>0$,

$$\|f(u)-f(v)\|\leq\kappa\|u-v\|\quad\text{for } u,v \text{ near } x.$$

In this case, the 'Clarke generalized derivative' $\partial^C f(x)$ is the convex hull of the set $\partial^L f(x)$ [16]. For concrete functions f, a convenient formula [5] is

$$\partial^C f(x)=\operatorname{cl}\operatorname{conv}\left\{\lim_r f'(u_r):u_r\to x,\ u_r\notin S\right\}$$

(for any zero-measure set S containing all points near x, where f is not differentiable). We call x 'Clarke critical' when $0\in\partial^C f(x)$.

The following simple result relates the above notions of critical points.

Proposition 1.4 (critical points) *The following implications hold for any continuous function f at any point*:

strongly critical $\Rightarrow$ *limiting critical*
$\Rightarrow$ *deformationally critical from below*
$\Rightarrow$ *Clarke critical*

(*where the final implication assumes f is locally Lipschitz*).

Proof The first implication follows directly from the definitions. To see the second implication, suppose the point x is limiting critical but deformationally regular from below. There is a sequence of points u_r approaching x and a sequence of Fréchet subgradients y_r approaching zero. Using the notation of Definition 1.1, for large indices r we have $\|u_r - x\| < \delta$ and $\|y_r\| < \sigma$, and hence as $t \downarrow 0$ we have

$$\begin{aligned} f(u_r) - \sigma t &\geq f(\phi(u_r, t)) \\ &\geq f(u_r) + y_r^T(\phi(u_r, t) - u_r) + o(\|\phi(u_r, t) - u_r\|) \\ &\geq f(u_r) - \|y_r\| \cdot \|\phi(u_r, t) - u_r\| + o(\|\phi(u_r, t) - u_r\|) \\ &\geq f(u_r) - \|y_r\| t + o(t). \end{aligned}$$

Thus $\|y_r\| \geq \sigma + o(1)$, which is a contradiction.

The third implication follows easily from well-known properties of the Clarke generalized gradient: if the point x is not Clarke critical, there exists a unit vector $w \in \mathbf{R}^n$ such that

$$\limsup_{u \to x,\ t \downarrow 0} \frac{f(u + tw) - f(u)}{t} < 0.$$

Now defining $\phi(u, t) = u + tw$ satisfies Definition 1.1, so x is deformationally regular from below. ■

As we have mentioned, any Morse critical point is also deformationally critical. But there is no definite connection between the concepts of Morse and limiting critical points. The previous example $t \mapsto t^3$ shows that a limiting critical point may not be Morse critical. Example 1.6 below shows the absence of the opposite implication: a Morse critical point that is not limiting critical.

Returning to the last proposition, we note that even for very simple functions, each of the converse implications in the proposition may fail. The function $f: \mathbf{R}^n \to \mathbf{R}$ is *piecewise linear* if there are polyhedral cones K_j and vectors a_j (for $j = 1, 2, \ldots, m$) such that $\cup_j K_j = \mathbf{R}^n$ and $f(x) = a_j^T x$ for all points $x \in K_j$.

Example 1.5 (limiting critical $\not\Rightarrow$ strongly critical) Consider the function $f: \mathbf{R}^2 \to \mathbf{R}$ defined by

$$f(y, z) = \left\{ \begin{array}{ll} |y| - |z| & \text{if } |y| \leq |z| \\ 0 & \text{otherwise} \end{array} \right\}.$$

The origin is not strongly critical, because $f(0, z) = -|z|$, but it is limiting critical because $f'(y, 0)$ is zero for all $y \neq 0$.

Example 1.6 (Morse critical $\not\Rightarrow$ limiting critical) Consider a function $f: \mathbf{R}^2 \to \mathbf{R}$ defined by $f(y, z) = |y| - |z|$. Now the origin is not limiting critical: a quick calculation shows $\partial^L f(0,0) = [-1, 1] \times \{-1, 1\}$. On the other hand, the level set $\{(y, z): f(y, z) < 0\}$ is not connected, and this implies, as we shall see, that the origin is Morse (and therefore deformationally) critical from below.

Example 1.7 (Clarke critical $\not\Rightarrow$ deformationally critical) Following Campa–Degiovanni [3], consider the function $f: \mathbf{R}^2 \to \mathbf{R}$ defined by $f(y, z) = |z - |y|| - y$. The Clarke generalized gradient at the origin is the square with corners $(-1 \pm 1, \pm 1)$, so the origin is Clarke critical. However, it is easy to check directly that the origin is not deformationally critical from below. Indeed, both the level sets $\{(y, z): f(y, z) > 0\}$ and $\{(y, z): f(y, z) < 0\}$ are connected and the set $\{(y, z): f(y, z) = 0\}$ has empty interior: as we shall see, this suffices to guarantee that the origin is deformationally regular from below.

2. Connected level sets

We refer to [9,13] for definitions and properties of o-minimal structures and definable sets and functions. The most prominent example is given by semi-algebraic sets and functions. A set $S \subset \mathbf{R}^n$ is 'semi-algebraic' if it is defined by some finite boolean combination of real polynomial inequalities and equations. A function $f: \mathbf{R}^n \to \mathbf{R}$ is 'semi-algebraic' if its graph (or equivalently, epigraph) is semi-algebraic: in particular, piecewise linear functions are semi-algebraic. We refer the reader to standard references such as [1,2] for semi-algebraic geometry.

Lemma 2.1 (connectedness of definable sets) *If the point x lies in the closed semi-algebraic set $S \subset \mathbf{R}^n$, then for all small $\epsilon > 0$, the set*

$$\{u \in S : \|u - x\| < \epsilon\}$$

is path-connected.

Proof The set $S \subset \mathbf{R}^n$ decomposes into a finite union of disjoint path-connected sets, each of which is also closed. The result then follows. ■

The following simple result describes a basic topological consequence of deformational regularity from below, and will suffice for our purposes. More sophisticated Morse-theoretic results appeared in [6,10].

Lemma 2.2 (path-connected level sets) *Consider a continuous function $f: \mathbf{R}^n \to \mathbf{R}$ that is Morse regular from below at the point $x \in \mathbf{R}^n$, and an open neighbourhood U of x. If the set $L_{\leq} = \{u \in U : f(u) \leq f(x)\}$ is path-connected, then so is the level set $L_{<} = \{u \in U : f(u) < f(x)\}$.*

Proof Consider any two points $u_1, u_2 \in L_{<}$. By assumption, for each index i, there is a continuous path $p_i : [0, 1] \to L_{\leq}$ satisfying $p_i(0) = x$ and $p_i(1) = u_i$. Using Definition 1.1, observe that for some $t_i > 0$ we have

$$\phi(p_i(s), t) \in U \quad \text{for all } s \in [0, 1],\ t \in [0, t_i].$$

Indeed, if this failed, then there would exists sequences $s^r \in [0, 1]$ and $t^r \downarrow 0$ satisfying $\phi(p_i(s^T), t^r) \notin U$. By continuity, any cluster point $\bar{s}$ of $\{s_r\}$ would then satisfy $p_i(\bar{s}) = \phi(p_i(\bar{s}, 0) \notin U$, which is a contradiction.

Let $t_3 = \min\{t_1, t_2\}$ and consider the following paths:

$$t \mapsto \phi(u_1, t), \quad \text{as } t \text{ increases from 0 to } t_3;$$
$$s \mapsto \phi(p_1(s), t_3), \quad \text{as } s \text{ decreases from 1 to 0;}$$
$$s \mapsto \phi(p_2(s), t_3), \quad \text{as } s \text{ increases from 0 to 1;}$$
$$t \mapsto \phi(u_2, t), \quad \text{as } t \text{ decreases from } t_3 \text{ to 0.}$$

These paths connect u_1 to $\phi(u_1, t_3), \phi(u_2, t_3)$, u_2, in turn, all in the set $L_{<}$, so the result follows. ■

The following result justifies our claim in Example 1.6 (Morse critical from below $\not\Rightarrow$ limiting critical).

PROPOSITION 2.3 (regular points and level sets) *Given a continuous function* $f: \mathbf{R}^n \to \mathbf{R}$, *consider a point* $x \in \mathbf{R}^n$ *that is Morse regular from below.*

(i) *If f is definable, then for all small* $\epsilon > 0$ *the set*

$$\{u \in \mathbf{R}^n : f(u) < f(x),\ \|u - x\| < \epsilon\}$$

is path-connected.

(ii) *If* $x = 0$ *and f is positively homogeneous, then the level set*

$$\{u : f(u) < 0\}$$

is path-connected.

Proof For the first part, apply Lemma 2.1 to the set $\{u \in \mathbf{R}^n : f(u) \le f(x)\}$ (which is closed and definable), followed by Lemma 2.2. For the second part, note that the level set $\{u\colon f(u) \le 0\}$ is obviously path-connected, so again we can apply Lemma 2.2. ■

3. Piecewise linear functions on the plane

The following is our technical result, on which the main theorems of the next section are based. Recall that a function $f: \mathbf{R}^n \to \mathbf{R}$ is 'piecewise linear' if there are polyhedral cones K_j and vectors a_j (for $j = 1, 2, \ldots, m$) such that $\cup_j K_j = \mathbf{R}^n$ and $f(x) = a_j^T x$ for all points $x \in K_j$.

THEOREM 3.1 (critical points of piecewise linear functions) *Consider a function* $f: \mathbf{R}^2 \to \mathbf{R}$ *that is continuous and positively homogeneous. If the origin is Morse regular from below, then the two open cones where f is strictly positive and strictly negative are both path-connected and their closures together cover* $\mathbf{R}^2$: *more precisely, there exist two angles* $\theta_1 < \theta_2$ *in the interval* $[0, 2\pi)$ *such that, in polar coordinates, we have*

$$f([r,\theta]) \begin{cases} < 0 & \text{if } \theta_1 < \theta < \theta_2 \\ = 0 & \text{if } \theta = \theta_1 \text{ or } \theta_2 \\ > 0 & \text{otherwise.} \end{cases}$$

If furthermore f is piecewise linear, then the converse is also true. Moreover, in this case zero is a deformationally regular point of f.

Proof The origin can be neither a local minimizer nor a local maximizer, by our observations in Section 1, so both the level sets $\{u\colon f(u) < 0\}$ and $\{u\colon f(u) > 0\}$ are nonempty, and the former is path-connected, by Proposition 2.3 (regular points and level sets).

Consider next any nonzero point u satisfying $f(u) = 0$. If u was a local minimizer, then by positive homogeneity, so would be the point λu, for all $\lambda > 0$. Thus each of these points would be Morse critical from below, contradicting the assumption that the origin is Morse regular from below. The claimed form for the level sets of f now follows.

To prove the second statement, we build the desired deformation. The function f is piecewise linear, so the set where its gradient is discontinuous is a finite union of rays generated by distinct points $x_0, x_1, \ldots, x_n$ on the unit circle. Without loss of generality, these points have arguments $0 = \theta_0 < \theta_1 < \cdots < \theta_n < 2\pi$ respectively, in polar coordinates. We denote the argument of any nonzero $u \in \mathbf{R}^2$ by $\arg u$. By adding more rays if necessary, we can suppose $\theta_{j+1} - \theta_j < \pi/2$ for each index j. By assumption, there exist vectors $a_0, a_1, \ldots, a_n \in \mathbf{R}^2$, such that, for each j,

$$f(x) = a_j^T x \quad \text{if } \theta_j \leq \arg x \leq \theta_{j+1},$$

where we define $\theta_{n+1} = 2\pi$ (and, correspondingly, $x_{n+1} = x_0$). In other words, we have covered the plane $\mathbf{R}^2$ by closed convex cones

$$K_j = \{x \neq 0 : \theta_j \leq \arg x \leq \theta_{j+1}\} \cup \{0\} = \mathbf{R}_+ x_j + \mathbf{R}_+ x_{j+1}$$

spanning an acute angle at the origin, and on each of which f is linear. After adding more rays if necessary, by our assumption on the level sets of f, we can suppose for some integer m we have

$$f(x_j) \begin{cases} = 0 & (j = 0, m) \\ > 0 & (0 < j < m) \\ < 0 & (m < j \leq n). \end{cases}$$

Notice, for each j we have

$$a_j^T x_j = f(x_j) \quad \text{and} \quad a_j^T x_{j+1} = f(x_{j+1}).$$

Consider next a number $\epsilon > 0$. Providing we choose ϵ sufficiently small, then, for $m < j < n$, the vector $h_j = x_{j+1} - \epsilon x_j$ satisfies

$$a_j^T h_j = f(x_{j+1}) - \epsilon f(x_j) < 0,$$

so the unit vector $g_j = |h_j|^{-1} h_j$ satisfies $a_j^T g_j < 0$. Notice that the half-line $x_j + \mathbf{R}_+ g_j$ intersects the ray $\mathbf{R}_+ x_{j+1}$: indeed, we have

$$x_j + \epsilon^{-1} \|h_j\| g_j = \epsilon^{-1} x_{j+1}. \tag{1}$$

Now define a (discontinuous) unit vector field F: $\mathbf{R}^2 \to \mathbf{R}^2$ by setting $F(0) = x_n$, and for nonzero $u \in \mathbf{R}^2$,

$$F(u) = \begin{cases} -x_1 & \text{if } \arg u > \theta_n \text{ or } \arg u \leq \theta_1 \\ -\|u\|^{-1} u & \text{if } \theta_1 \leq \arg\ u \leq \theta_{m-1} \\ -x_{m-1} & \text{if } \theta_{m-1} \leq \arg\ u < \theta_{m+1} \\ g_j & \text{if } \theta_j \leq \arg\ u < \theta_{j+1} \text{ for } m < j < n \\ x_n & \text{if } \arg\ u = \theta_n. \end{cases}$$

Then our desired deformation $\phi : \mathbf{R}^2 \times \mathbf{R}_+ \to \mathbf{R}^2$ is defined uniquely by the property that, for each point $u \in \mathbf{R}^2$, the trajectory $t \mapsto \phi(u, t)$ solves the initial value problem

$$\frac{\mathrm{d}z}{\mathrm{d}t} = F(z), \quad z(0) = u \tag{2}$$

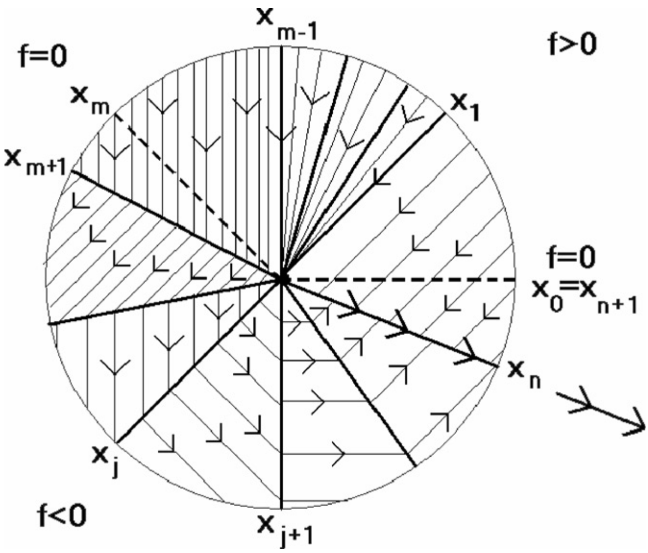

Figure 1. Deformation trajectories inside the unit disk.

(where henceforth derivatives with respect to t are taken on the right). In other words, from any initial point u, the point $\phi(u, t)$ follows at unit speed a piecewise linear trajectory whose direction at any point is given by the field (Figure 1).

Proving the required properties of the deformation is elementary, if a little lengthy. Arguing from the last observation, the first desired property of the deformation,

$$\|\phi(u,t) - u\| \leq t, \quad \text{for all } u \in \mathbf{R}^2, \ t \geq 0,$$

follows immediately. To check the second desired property, namely

$$f(\phi(u,t)) \leq f(u) - \sigma t \tag{3}$$

for some $\sigma > 0$, notice

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\phi(u,t)) = f'\left(\phi(u,t); \frac{\mathrm{d}}{\mathrm{d}t}\phi(u,t)\right) = f'(\phi(u,t); F(\phi(u,t)))$$

(where $f'(\cdot;\cdot)$ denotes the directional derivative). We therefore need to check that $f'(u, F(u)) < -\sigma$ for all points $u \in \mathbf{R}^2$. We consider the five cases in the definition of the field F in turn: clearly it suffices to show that $f'(u, F(u))$ is uniformly negative in each case.

Case 1 Suppose $u \in K_n \backslash \mathbf{R}_+ x_n$. By assumption, we can write $x_0 = \lambda x_1 + \mu x_n$ with $\lambda, \mu > 0$. Clearly $u - tx_1 \in K_n$ for all small $t \geq 0$, so

$$f'(u, F(u)) = -a_n^T x_1 = -a_n^T \frac{1}{\lambda}(x_0 - \mu x_n) = \frac{\mu}{\lambda} f(x_n) < 0.$$

On the other hand, suppose that $u \in \operatorname{int} K_0$. Then

$$f'(u, F(u)) = -a_0^T x_1 = -f(x_1) < 0.$$

Case 2 Suppose $0 \neq u \in K_j$ where $1 \leq j \leq m-2$. We can write $u = \lambda x_j + \mu x_{j+1}$ for some λ, $\mu \geq 0$ not both zero. Clearly $u - t\|u\|^{-1} \in K_j$ for all small $t \geq 0$, so by convexity,

$$\begin{aligned} f'(u, F(u)) &= -a_j^T \|u\|^{-1} u \\ &= -a_j^T \left\| \lambda x_j + \mu x_{j+1} \right\|^{-1} \left(\lambda x_j + \mu x_{j+1} \right) \\ &\leq -\min\{f(x_j), f(x_{j+1})\} \left\| \frac{\lambda}{\lambda+\mu} x_j + \frac{\mu}{\lambda+\mu} x_{j+1} \right\|^{-1} \\ &\leq -\min\{f(x_j), f(x_{j+1})\} \\ &< 0. \end{aligned}$$

Case 3 If $u \in (\mathrm{int}\, K_{m-1}) \cup (K_m \backslash \mathbf{R}_+ x_{m+1})$, the argument is completely analogous to Case 1.

Case 4 Suppose $u \in K_j \backslash \mathbf{R}_+ x_{j+1}$, where $m < j < n$. Clearly $u + t x_j \in K_j$ for all small $t \geq 0$, so

$$f'(u, F(u)) = a_j^T g_j < 0$$

by our choice of the number ϵ.

Case 5 If $u \in \mathbf{R}_+ x_n$, then

$$f'(u, F(u)) = a_n^T x_n = f(x_n) < 0.$$

This completes the proof of property (3).

It remains to prove that the deformation ϕ is continuous. Since $\|F(u)\| = 1$ for all u, we immediately deduce that $\phi(u, \cdot)$ has Lipschitz constant one for any fixed u:

$$\|\phi(u,s) - \phi(u,t)\| \leq |s - t| \quad \text{for all } s, t \geq 0.$$

It therefore suffices to show that $\phi(\cdot, t_0)$ is continuous for any fixed $t_0 \geq 0$.

Notice that ϕ is positively homogeneous:

$$\phi(\alpha u, \alpha t) = \alpha \phi(u, t) \quad \text{for all } u \in \mathbf{R}^2,\ t, \alpha \geq 0.$$

To see this, without loss of generality suppose $\alpha > 0$ and define a new function ψ: $\mathbf{R}^2 \times \mathbf{R}_+ \to \mathbf{R}^2$ by

$$\psi(u, t) = \frac{1}{\alpha} \phi(\alpha u, \alpha t).$$

Then $\psi(u, 0) = u$ for all $u \in \mathbf{R}^2$, and

$$\frac{\mathrm{d}}{\mathrm{d}t} \psi(u, t) = \phi_t(\alpha u, \alpha t) = F(\phi(\alpha u, \alpha t)) = F(\alpha \psi(u, t)) = F(\psi(u, t))$$

so ψ solves the initial value problem (2), and hence coincides with ϕ.

The restriction of the function $\phi(\cdot, t_0)$ to the set $\cup_{j=1}^{m-2} K_j$ is continuous, since for points u in this set we have

$$\phi(u, t_0) = \begin{cases} (1 - \|u\|^{-1} t_0) u & (\|u\| \geq t_0) \\ (t_0 - \|u\|) x_n & (\|u\| \leq t_0). \end{cases}$$

We next prove ϕ is continuous relative to the set $U \times \mathbf{R}_+$, where

$$U = \{u \in \mathbf{R}^2 : u \neq 0,\ \theta_{m-1} < \arg\ u \leq \theta_n\}.$$

To see this, we consider the trajectory passing through the point x_{m+1}. More precisely, consider the continuous function γ: $\mathbf{R} \to U$ defined by

$$\gamma(t) = \begin{cases} x_{m+1} - t x_{m-1} & (t \leq 0) \\ \phi(x_{m+1}, t) & (t \geq 0). \end{cases}$$

By definition, for all points $u \in U$ *we* know arg $F(u) \in$ arg $u + (0, \pi)$ and $|F(u)| = 1$. Since $\gamma'(t) = F(\gamma(t))$, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t} \arg \gamma(t) > 0 \quad \text{whenever } \gamma(t) \in U.$$

Using Equation (1), we see that the trajectory $\gamma(t)$ hits the ray $\mathbf{R}_+ x_n$ at time

$$\bar{t} = \frac{1}{\epsilon} \sum_{j=m+1}^{n-1} \|h_j\|.$$

So, on the interval $(-\infty, t]$, the function $t \mapsto \arg \gamma(t)$ is strictly increasing, with

$$\lim_{t \downarrow -\infty} \arg \gamma(t) = \theta_{m-1} \text{ and } \arg \gamma(\bar{t}) = \theta_n,$$

so there is a continuous, strictly increasing inverse function

$$\xi : (\theta_{m-1}, \theta_n] \to (-\infty, \bar{t}], \quad \text{with } \arg \gamma(\xi(\theta)) = \theta\ (\theta_{m-1} < \theta \leq \theta_n).$$

Any point $u \in U$ therefore satisfies arg $\gamma(\xi(\arg u)) = \arg u$, so

$$u = \frac{\|u\|}{\|\gamma(\xi(\arg u))\|} \gamma(\xi(\arg u)).$$

Now we can use positive homogeneity to see, for all $(u, t) \in U \times \mathbf{R}_+$, the relationship

$$\begin{aligned} \phi(u, t) &= \frac{\|u\|}{\|\gamma(\xi(\arg u))\|} \phi\left(\gamma(\xi(\arg u)), \frac{\|\gamma(\xi(\arg u))\|}{\|u\|} t\right) \\ &= \frac{\|u\|}{\|\gamma(\xi(\arg u))\|} \gamma\left(\xi(\arg u) + \frac{\|\gamma(\xi(\arg u))\|}{\|u\|} t\right). \end{aligned}$$

The right-hand side is clearly continuous, so our claim follows. A similar, easier argument shows that ϕ is also continuous relative to the set

$$(\mathbf{R}_{++} x_n \cup \operatorname{int}(K_n \cup K_0)) \times \mathbf{R}_+.$$

It remains only to show that $\phi(\cdot, t_0)$ is continuous at any point on the rays $\mathbf{R}_+ x_1$ and $\mathbf{R}_+ x_m$. By positive homogeneity, we can focus on the points x_m (the argument for x_1 being similar) and zero. The following lemma is helpful.

LEMMA 3.2 *Consider sequences of numbers $p_r \to +\infty$ and $\{q_r\}$ and any $t \geq 0$. Then we have*

$$\frac{q_r}{p_r} \to 1 \Rightarrow \frac{1}{p_r}\gamma(p_r t - q_r) \to \phi(x_{m-1}, t)$$
$$\frac{q_r}{p_r} \to 0 \Rightarrow \frac{1}{p_r}\gamma(p_r t - q_r) \to \phi(0, t).$$

Proof By definition, we have

$$\phi(x_{m-1}, t) = \begin{cases} (1-t)x_{m-1} & \text{if } t \leq 1 \\ (t-1)x_n & \text{if } t \geq 1, \end{cases}$$
$$\phi(0, t) = tx_n.$$

On the other hand,

$$\gamma(p_r t - q_r) \begin{cases} = x_{m+1} - (q_r - p_r t)x_{m-1} & \text{if } t \leq \dfrac{q_r}{p_r} \\ = \gamma(\bar{t}) + (p_r t - q_r - \bar{t})x_n & \text{if } t \geq \dfrac{(q_r + \bar{t})}{p_r} \\ \in \gamma[0, \bar{t}] & \text{otherwise.} \end{cases}$$

Dividing through by p_r and taking the limit, noting that the path $\gamma[0, \bar{t}]$ is bounded, now proves the result. ■

Proceeding with the proof, consider now a sequence of points $u_r \to x_{m-1}$. We want to show $\phi(u_r, t) \to \phi(x_{m-1}, t)$, for any $t \geq 0$. By considering subsequences, we can suppose either that $u_r \in K_{m-2}$ for all r or that $u_r \in U$ for all r. The first case is already proved above, so consider the second case. As above, we then have

$$\phi(u_r, t) = \frac{\|u_r\|}{\|\gamma(\xi(\arg u_r))\|}\gamma\left(\xi(\arg u_r) + \frac{\|\gamma(\xi(\arg u_r))\|}{\|u_r\|}t\right). \tag{4}$$

Since $u_r \to x_{m-1}$, we have $\theta_{m-1} < \arg\, u_r \to \theta_{m-1}$, so $\xi(\arg\, u_r) \to -\infty$. Notice

$$\frac{\gamma(-s)}{s} \to x_{m-1} \quad \text{and} \quad \frac{\gamma(s)}{s} \to x_n \text{ as } s \to \infty. \tag{5}$$

If we now apply Lemma 3.2 with

$$p_r = \frac{\|\gamma(\xi(\arg u_r))\|}{\|u_r\|} \quad \text{and} \quad q_r = -\xi(\arg u_r), \tag{6}$$

our desired result follows.

Finally, consider a sequence of nonzero points $u_r \to 0$. We want to show $\phi(u_r, t) \to \phi(0, t)$, for any $t \geq 0$. By considering subsequences, we can restrict to three cases: $\theta_1 \leq \arg\, u_r \leq \theta_{m-1}$ for all r, or $u_r \in U$ for all r, or $\arg\, u_r \in (\theta_n, 2\pi) \cup [0, \theta_1)$ for all r. The first case is proved above. On the other hand the second and third involve similar arguments, so suppose $u_r \in U$ for all r.

Using Equations (4) and (6) again, we have

$$\phi(u_r, t) = \frac{1}{p_r}\gamma(p_r t - q_r).$$

Property (5) and the fact that $\gamma(s) \neq 0$ for all $s \in \mathbf{R}$ guarantees that $p_r \to \infty$, and, furthermore, that $s/\|\gamma(s)\|$ is bounded above, so $q_r/p_r \to 0$. The desired result now follows by Lemma 3.2. ■

4. Main theorems

Since the notion of deformationally regular from below is purely local, we immediately arrive at the following characterization for continuous 'piecewise affine' functions of two variables (from which we understand a decomposition of $\mathbf{R}^2$ into a finite disjoint union of points, line segments and open polygons on each of which the function f agrees with an affine function).

THEOREM 4.1 (piecewise affine functions on $\mathbf{R}^2$) *Suppose the continuous function $f: \mathbf{R}^2 \to \mathbf{R}$ is piecewise affine. Then a point $\bar{x} \in \mathbf{R}^2$ is deformationally regular from below if and only if, for all small $\delta > 0$, the strict local level sets*

$$\{x \in B_\delta(\bar{x}) : f(x) < f(\bar{x})\} \quad \textit{and} \quad \{x \in B_\delta(\bar{x}) : f(x) > f(\bar{x})\}$$

are both path-connected and their closures together cover the neighbourhood $B_\delta(\bar{x})$.

We can push the basic result of Theorem 3.1 in a different direction, for definable functions.

THEOREM 4.2 (definable functions on the plane) *Suppose the definable function $f: \mathbf{R}^2 \to \mathbf{R}$ is continuous near $\bar{x}$. Then $\bar{x}$ is Morse regular from below if and only if there is a $\delta > 0$ such that the strict local level sets*

$$\{x \in B_\delta(\bar{x}) : f(x) < f(\bar{x})\} \quad \textit{and} \quad \{x \in B_\delta(\bar{x}) : f(x) > f(\bar{x})\}$$

are both path-connected, and their closures together cover a neighbourhood of $\bar{x}$.

Proof Again we only have to prove that the conditions imply Morse regularity. We can assume of course that $\bar{x} = 0$ and $f(\bar{x}) = 0$. It suffices to show the existence of a definable homeomorphism Φ of a neighbourhood of zero in $\mathbf{R}^2$ onto a further neighbourhood of zero such that $\Phi(0) = 0$ and the restriction of $g = f \circ \Phi$ on a small neighbourhood of zero is piecewise linear. In this case the sets, extending g by homogeneity to all of $\mathbf{R}^2$, we see that the sets $\{x: g(x) < 0\}$ and $\{x: g(x) > 0\}$ are path-connected. Indeed, suppose $g(x_i) < 0$, for $i = 1, 2$. Then for some $\lambda > 0$ and $z_i = \Phi(\lambda x_i)$ we have $\|z_i\| < \delta$ and $f(z_i) = g(\lambda x_i) < 0$. By assumption there is a continuous path $z(t)$ joining z_1 and z_2 and such that $\|z(t)\| < \delta$ and $f(z(t) < 0$ for all t. Then $x(t) = \Phi^{-1}(z(t)$ is a continuous path joining x_1 and x_2 and completely lying in $\{x: g(x) < 0\}$.

Next, consider any x such that $\|\Phi(x)\| < \delta$. Then $z = \Phi(x)$ belongs to the closure of either $\{z: f(z) < 0\}$ or $\{z: f(z) > 0\}$. But then x must belong to the closure of the image of one of these sets under Φ^{-1}, that is, to the closure of either $\{x: g(x) < 0\}$ or $\{x: g(x) > 0\}$.

Thus g satisfies all assumptions of Theorem 3.1, which means that zero is a regular point of g. Let ϕ be the corresponding deformation, that is, for some $\sigma > 0$, we have

$$g(u) - g(\phi(u, t)) \geq \sigma t$$

for all small u and small $t \geq 0$. Set $\psi(v, t) = \Phi(\phi(\Phi^{-1}(v), t))$. Then

$$f(v) - f(\psi(v, t)) = g(\Phi^{-1}(v)) - g(\phi(\Phi^{-1}(v)), t \geq \sigma t$$

and we conclude that zero is a Morse regular point of f as claimed.

Thus, we need to prove the existence of the desired homeomorphism. To this end we first recall some basic facts concerning simplices and simplicial complexes in $\mathbf{R}^n$.

A simplex S in $\mathbf{R}^n$ of dimension $k \le n$ is the convex hull of $k+1$ affinely independent points $x_1, \ldots, x_{k+1}$ (that is, such that none of them is a convex combination of the others) called *vertices* of the simplex. A simplex whose vertices form a proper subset of $\{x_1, \ldots, x_{k+1}\}$ is a *face* of S. A finite collection $\Sigma = \{S_i\}$ is a 'simplicial complex' if faces of any element of Σ also belong to σ. The union of all elements of Σ is often called the body of Σ. We shall not make a distinction between a simplex and its body and shall use the same symbol for both. (Observe that in our definition simplices and complexes are closed sets.)

The basic fact we need to construct a desired homeomorphism is the following triangulation theorem [9, Theorem 4.5]. ■

THEOREM 4.3 (triangulation theorem) *Let f be a continuous definable function on $\mathbf{R}^n$ with a bounded domain and closed graph. Then there is a simplicial complex $\Sigma \subset \mathbf{R}^{n+1}$ and a definable homeomorphism Ψ: $\Sigma \to \mathbf{R}^n$ onto the domain of f such that $g = f \circ \Psi$ is a piecewise affine function.*

More precisely the theorem says that the restriction of g to any simplex of Σ is an affine function.

Return to our proof. Taking, if necessary, the restriction of f to a closed bounded neighbourhood of zero, we can guarantee that the graph of f is closed, so the triangulation theorem can be applied. Let $\Sigma \subset \mathbf{R}^3$ and Ψ be the corresponding complex and homeomorphism. We can assume that $0 \in \Sigma$ and $\Psi(0) = 0$.

Observe that the triangulation theorem is a global result. What we need is a local homeomorphism but from a neighbourhood of zero in $\mathbf{R}^2$ rather than from a complex in $\mathbf{R}^3$.

As we need a homeomorphism of a small neighbourhood of zero, we are interested in simplices of Σ containing zero. There is no loss of generality in assuming that zero is a vertex of any of them. Let $l_1, \ldots, l_k$ be one dimensional simplices containing zero, that is $l_i = [0, x_i]$, and let $S_1, \ldots, S_m$ the two-dimensional simplices containing zero. By definition, every l_i is a face of a certain S_j.

We claim that actually every l_i is a face of exactly two simplices S_j. Indeed, assume, e.g. that l_1 is a face of S_1 only. Then $\Psi(l_1)$ belongs to the boundary of $\Psi(S_1)$. On the other hand, for a small $\lambda > 0$ the point $\Psi(\lambda x_1)$ belongs to the interior of the domain of f and hence of $\Psi(\Sigma)$. This means that there is a sequence $(u_k) \subset \Sigma \backslash S_1$ such that $\Psi(u_k)$ converge to λx_1. We may assume that all u_k belong to some S_j, $j \neq 1$ and therefore there is a $u \in S_j$ such that $\Psi(u) = \lambda x_1$, in contradiction with the fact that Ψ is one-to-one. It is also clear that no l_i can be a face of three two-dimensional simplices (in which case, any relative interior point of l_i would have a small neighbourhood whose intersection with Σ would contain a disc with a half disc glued to part of its – diameter – a configuration that cannot be homeomorphic to a domain in $\mathbf{R}^2$).

Thus we have to conclude that $k = m$ and by renumbering the simplices we may assume that l_i and $l_{i+1}(\text{mod } k)$ are faces of S_i for $i = 1, \ldots, k$. Now we can easily conclude the proof.

Let us divide the unit circle in $\mathbf{R}^2$ into k equal parts. Let $x_1, \ldots, x_k$ be the points of the partition. Consider the following mapping Γ from $\mathbf{R}^2$ to $\mathbf{R}^3$:

$$\Gamma(0) = 0, \Gamma(x_i) = z_i, \quad (i = 1, 2, \ldots, k)$$

and $\Gamma(\cdot)$ is a linear mapping on every cone generated by pairs of adjacent x_i.

Then $\Phi = \Psi \circ \Gamma$ is the desired homeomorphism.

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