THE LAX CONJECTURE IS TRUE

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ABSTRACT. In 1958 Lax conjectured that hyperbolic polynomials in three variables are determinants of linear combinations of three symmetric matrices. This conjecture is equivalent to a recent observation of Helton and Vinnikov.

Consider a polynomial p on $\mathbf{R}^n$ of degree d (the maximum of the degrees of the monomials in the expansion of p). We call p *homogeneous* if $p(tw) = t^d p(w)$ for all real t and vectors $w \in \mathbf{R}^n$: equivalently, every monomial in the expansion of p has degree d. We denote the set of such polynomials by $\mathbf{H}^n(d)$. By identifying a polynomial with its vector of coefficients, we can consider $\mathbf{H}^n(d)$ as a normed vector space of dimension $\binom{n+d-1}{d}$.

A polynomial $p \in \mathbf{H}^n(d)$ is *hyperbolic* with respect to a vector $e \in \mathbf{R}^n$ if $p(e) \neq 0$ and, for all vectors $w \in \mathbf{R}^n$, the univariate polynomial $t \mapsto p(w - te)$ has all real roots. The corresponding *hyperbolicity cone* is the open convex cone (see [5])

$$\{w \in \mathbf{R}^n : p(w - te) = 0 \ \Rightarrow \ t > 0\}.$$

For example, the polynomial $w_1 w_2 \cdots w_n$ is hyperbolic with respect to the vector $(1, 1, \ldots, 1)$, since the polynomial $t \mapsto (w_1 - t)(w_2 - t) \cdots (w_n - t)$ has roots $w_1, w_2, \ldots, w_n$; hence the corresponding hyperbolicity cone is the open positive orthant.

Hyperbolic polynomials and their hyperbolicity cones originally appeared in the partial differential equations literature [4]. They have attracted attention more recently as fundamental objects in modern convex optimization [6, 1]. Three primary reasons drive this interest:

  (i) the definition of "hyperbolic polynomial" is strikingly simple;
  (ii) the class of hyperbolic polynomials, although not well-understood, is known to be rich — specifically, its interior in $\mathbf{H}^n(d)$ is nonempty;
  (iii) optimization problems posed over hyperbolicity cones, with linear objective and constraint functions, are amenable to efficient interior point algorithms.

For more details on these reasons, see [6, 1].

In light of the interest of hyperbolic polynomials to optimization theorists, it is therefore natural to ask: how general is the class of hyperbolicity cones? In particular, do hyperbolicity cones provide a more general model for convex optimization

than "semidefinite programming" (the study of optimization problems with linear objectives and constraints and semidefinite matrix variables [9])?

We begin with some easy observations. A rich source of examples of hyperbolicity cones are *semidefinite slices*, by which we mean sets of the form

$$\text{(1)} \qquad \left\{ w : \sum_{j=1}^{n} w_j G_j \in \mathbf{S}_{++}^d \right\},$$

for matrices $G_1, G_2, \ldots, G_n$ in the space $\mathbf{S}^d$ of all d-by-d real symmetric matrices, where $\mathbf{S}_{++}^d$ denotes the positive definite cone. Such cones are, in particular, "semidefinite representable" in the sense of [9].

Proposition 2. *Any nonempty semidefinite slice is a hyperbolicity cone.*

Proof. Suppose the semidefinite slice (1) contains the vector $\widehat{w}$. We claim the polynomial p on $\mathbf{R}^n$ defined by

$$\text{(3)} \qquad p(w) = \det \sum_j w_j G_j$$

is hyperbolic with respect to $\widehat{w}$, with corresponding hyperbolicity cone described by (1). Clearly p is homogeneous of degree d, and $p(\widehat{w}) > 0$.

Define a matrix $\widehat{G} = \sum_j \widehat{w}_j G_j \in \mathbf{S}_{++}^d$, and notice, for any vector $w \in \mathbf{R}^n$ and scalar t, we have

$$p(w - t\widehat{w}) = \det \sum_j (w_j - t\widehat{w}_j)G_j = \det \left(\sum_j w_j G_j - t\widehat{G} \right)$$
$$= (\det \widehat{G}) \det \left(\widehat{G}^{-1/2} \Big[\sum_j w_j G_j \Big] \widehat{G}^{-1/2} - tI \right),$$

where I denotes the identity matrix. Consequently, the univariate polynomial $t \mapsto p(w - t\widehat{w})$ has all real roots, namely the eigenvalues of the symmetric matrix $H = \widehat{G}^{-1/2}[\sum_j w_j G_j]\widehat{G}^{-1/2}$, so p is hyperbolic with respect to $\widehat{w}$. Furthermore, by definition, w lies in the corresponding hyperbolicity cone exactly when these roots (or equivalently, eigenvalues) are all strictly positive. But this property is equivalent to H being positive definite, which holds if and only if $\sum_j w_j G_j$ is positive definite, as required. $\qquad \square$

The class of semidefinite slices is quite broad. For example, any *homogeneous cone* (an open convex pointed cone whose automorphism group acts transitively) is a semidefinite slice [2] (see also [3]). In particular, therefore, any homogeneous cone is a hyperbolicity cone, a result first observed in [6].

What about the converse? When is a hyperbolicity cone a semidefinite slice? How general is the class of hyperbolic polynomials of the form (3)?

In considering a general hyperbolic polynomial p on $\mathbf{R}^n$ with respect to a vector e, we can suppose, after a change of variables, that $e = (1, 0, 0, \ldots, 0)$ and $p(e) = 1$. Consider the first nontrivial case, that of $n = 2$. By assumption, the polynomial $t \mapsto p(-t, 1)$ has all real roots, which we denote $g_1, g_2, \ldots, g_d$, so for some nonzero real k we have the identity

$$p(-t, 1) = k \prod_{j=1}^{d} (g_j - t).$$

By homogeneity, for any vector $(x, y) \in \mathbf{R}^2$ with $y \neq 0$, we deduce

$$p(x, y) = y^d p\left(\frac{x}{y}, 1\right) = y^d k \prod_{j=1}^{d} \left(g_j + \frac{x}{y}\right) = k \prod_{j=1}^{d} (g_j y + x).$$

By continuity and the fact that $p(1, 0) = 1$, we see that

$$p(x, y) = \prod_{j=1}^{d} (g_j y + x) = \det(xI + yG)$$

for all $(x, y) \in \mathbf{R}^2$, where G is the diagonal matrix with diagonal entries $g_1, g_2, \ldots, g_d$. Thus any such hyperbolic polynomial p does indeed have the form (3).

What about hyperbolic polynomials in more than two variables? The following conjecture [8] proposes that all hyperbolic polynomials in three variables are likewise easily described in terms of determinants of symmetric matrices.

Conjecture 4 (Lax, 1958). *A polynomial p on $\mathbf{R}^3$ is hyperbolic of degree d with respect to the vector $e = (1, 0, 0)$ and satisfies $p(e) = 1$ if and only if there exist matrices $B, C \in \mathbf{S}^d$ such that p is given by*

(5) $$p(x, y, z) = \det(xI + yB + zC).$$

An obvious consequence of this conjecture would be that, in $\mathbf{R}^3$, hyperbolicity cones and semidefinite slices comprise identical classes.

A polynomial on $\mathbf{R}^2$ is a *real zero polynomial* [7] if, for all vectors $(y, z) \in \mathbf{R}^2$, the univariate polynomial $t \mapsto q(ty, tz)$ has all real roots. Such polynomials are closely related to hyperbolic polynomials via the following elementary result.

Proposition 6. *If p is a hyperbolic polynomial of degree d on $\mathbf{R}^3$ with respect to the vector $e = (1, 0, 0)$, and $p(e) = 1$, then the polynomial on $\mathbf{R}^2$ defined by $q(y, z) = p(1, y, z)$ is a real zero polynomial of degree no more than d, and satisfying $q(0, 0) = 1$.*

Conversely, if q is a real zero polynomial of degree d on $\mathbf{R}^2$ satisfying $q(0, 0) = 1$, then the polynomial on $\mathbf{R}^3$ defined by

(7) $$p(x, y, z) = x^d q\left(\frac{y}{x}, \frac{z}{x}\right) \quad (x \neq 0)$$

(extended to $\mathbf{R}^3$ by continuity) is a hyperbolic polynomial of degree d on $\mathbf{R}^3$ with respect to e, and $p(e) = 1$.

Proof. To prove the first statement, note that for any point $(y, z) \in \mathbf{R}^2$ and complex μ, if $q(\mu(y, z)) = 0$, then $\mu \neq 0$ and $0 = p(1, \mu y, \mu z) = \mu^d p(\mu^{-1}, y, z)$, using the homogeneity of p. So, by the hyperbolic property, $-\mu^{-1}$ is real, and hence so is μ. The remaining claims are clear.

For the converse direction, since q has degree d, clearly p is well-defined and homogeneous of degree d and satisfies $p(e) = 1$. If $p(\mu, y, z) = 0$, then either $\mu = 0$ or $q(\mu^{-1}(y, z)) = 0$, in which case μ^{-1} and hence also μ must be real. $\square$

(Notice, in the first claim of the proposition, that the polynomial q may have degree strictly less than d: consider, for example, the case $p(x, y, z) = x^d$.)

Helton and Vinnikov [7, p. 10] observe the following result, based heavily on [10].

Theorem 8. *A polynomial q on $\mathbf{R}^2$ is a real zero polynomial of degree d and satisfies $q(0,0) = 1$ if and only if there exist matrices $B, C \in \mathbf{S}^d$ such that q is given by*

$$(9) \qquad\qquad q(y,z) = \det(I + yB + zC).$$

(Notice, as in the Lax conjecture, that the "if" direction is immediate.)

We claim that Theorem 8 is equivalent to the Lax conjecture. To see this, suppose p is a hyperbolic polynomial of degree d on $\mathbf{R}^3$ with respect to the vector $e = (1,0,0)$, and $p(e) = 1$. Then by Proposition 6, the polynomial on $\mathbf{R}^2$ defined by $q(y,z) = p(1,y,z)$ is a real zero polynomial of degree $d' \leq d$, and satisfying $q(0,0) = 1$. Hence by Theorem 8, equation (9) holds: we can assume $d' = d$ by replacing $B, C \in \mathbf{S}^{d'}$ with block diagonal matrices $\mathrm{Diag}(B,0), \mathrm{Diag}(C,0) \in \mathbf{S}^d$. Then, by homogeneity, for $x \neq 0$,

$$p(x,y,z) = x^d p\left(1, \frac{y}{x}, \frac{z}{x}\right) = x^d q\left(\frac{y}{x}, \frac{z}{x}\right)$$
$$= x^d \det\left(I + \frac{y}{x}B + \frac{z}{x}C\right) = \det(xI + yB + zC),$$

as required. The converse direction in the Lax conjecture is immediate.

Conversely, let us assume the Lax conjecture, and suppose q is a real zero polynomial of degree d on $\mathbf{R}^2$ satisfying $q(0,0) = 1$. (The converse direction in Theorem 8 is immediate.) Then by Proposition 6 the polynomial p defined by equation (7) is a hyperbolic polynomial of degree d on $\mathbf{R}^3$ with respect to e, and $p(e) = 1$. According to the Lax conjecture, equation (5) holds, so

$$q(y,z) = p(1,y,z) = \det(I + yB + zC),$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The exact analogue of the Lax conjecture fails in general for polynomials in $n > 3$ variables. To see this, note that the set of polynomials on $\mathbf{R}^n$ of the form $w \mapsto \det \sum_j w_j G_j$ (where $G_1, G_2, \ldots, G_n \in \mathbf{S}^d$) has dimension at most $n \cdot \binom{d+1}{2}$, being an algebraic image of a vector space of this dimension. If the degree d is large, this dimension is certainly smaller than the dimension of the set of hyperbolic polynomials: as we observed above, this latter set has nonempty interior in the space $\mathbf{H}^n(d)$ (by a result of Nuij [6, Thm. 2.1]), and so has dimension $\binom{n+d-1}{d}$.

More concretely, consider the polynomial defined by $p(w) = w_1^2 - \sum_2^n w_j^2$ for $w \in \mathbf{R}^n$. This polynomial is hyperbolic of degree $d = 2$ with respect to the vector $(1, 0, 0, \ldots, 0)$, and yet cannot be written in the form $\det \sum_j w_j G_j$ for matrices $G_1, G_2, \ldots, G_n \in \mathbf{S}^2$ if $n > 3$. To see this, choose any nonzero vector w satisfying $w_1 = 0$, and such that the first row of the matrix $\sum_j w_j G_j$ is zero.

The question of whether all hyperbolicity cones are semidefinite slices, or, more generally, are semidefinite representable, appears open.

Acknowledgement

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